A New Short TREATISE OF Algebra:

With the Geometrical Construction of

EQUATIONS,

As far as the

Fourth *Power* or *Dimension.*

Together with a Specimen of the

NATURE and ALGORITHM OF FLUXIONS.

The Third Edition with Additions.

By *JOHN HARRIS*, D. D. and Fellow of the *Royal Society*.

LONDON:
Printed for Dan. Midwinter, at the *Three Crowns* in St. *Paul's* Church-Yard. MDCCXIV.

THE

Epiſtle Dedicatory

TO THE

Learned and Ingenious

JOHN BRIDGES Eſq;

Sollicitor of the Cuſtoms.

Dear SIR,

WHEN I conſider the ancient Date of our Friendſhip, and reflect, as I always do with Pleaſure, on the many happy Hours we have ſpent together, and that too in Diſcourſes we need not now be aſham'd of: When I ſee ſtill in my Friend the ſame true *Engliſh* Honeſty and Integrity, the ſame ſincere Love for Virtue and Honour, and the ſame warm Affection for all

all useful and substantial Learning, which at first made me justly Admire him and Devote my self to him: When I consider a Genius every way improv'd (and no way injur'd) by Travels into Countries, from whence too many bring home nothing but Vice and Impertinence: And when I am so happy as to find by Experience, that even the hurry of Business it self cannot make Mr. *Bridges* forget or neglect either Learning or his Friend: Then would I gladly shew that I am neither Unjust nor Ungrateful, but as doubly a Debtor both to your Merit and to your Friendship, tell the World I have a due Sense of both: Permit me then, *Dear Sir!* to do it this way, and to continue, as I have long done,

Your Real Friend, and

most Obliged Humble Servant,

John Harris.

TO

TO THE READER.

THIS *ſmall Tract of that Admirable Science*, Algebra, *was written primarily for the Uſe of my Auditors at the* Publick Mathematick Lecture; *which was ſet up at the* Marine Coffee-Houſe *in* Birchin-Lane, *entirely for the Publick Good, by the Generous* Charles Cox *Eſq; Member of Parliament for the Burgh of* Southwark.

The Book is ſhort indeed, but I think plain enough every where, eſpecially in thoſe Parts which have been leſs Treated of in our own Language, viz. The Geometrical Conſtruction of Equations: *Which I have carried as far as to* Biquadraticks, *and have ſhew'd you plainly how to Conſtruct all Equations not exceeding four Dimenſions, by the help of a Circle interſecting the Curve of a* Parabola.

I have alſo demonſtrated the Properties of the Parabola *on which thoſe Conſtructions depend; and have given you beſides, the Method of the Inveſtigation of the famous Mr.* Baker's Central Rule.

And

And because I have not yet found it done by any one in the English *Tongue, I have given you a short Account of the Nature and Algorithm of* Fluxions, *and one or two Instances of their Use and Application: But in this I have been designedly as brief as possibly I could, intending by it only to stir up the Reader's Curiosity to peruse those Excellent Treatises which I have there mentioned: And which when he hath done, I know I shall have his Thanks for that little* Sketch *of* Fluxions, *which he will find there.*

N. B. Mr. *Hayes's* Book was not then Published.

In the whole I have proposed to my self rather to Instruct *the Young* English *Beginner, than to* Improve *the Learned Proficient; as knowing that if I can but help to lay a good substantial Foundation for him, his own Diligence and Application will raise the Structure to what height he pleases; but without beginning right (which is too commonly neglected) nothing is to be done.*

In the Second Edition *I endeavoured to Correct the Faults of the Former; and added some few things in the* Fluxions, *to make that Matter as Plain and Intelligible as I could, in so little a room.*

And to those who would pursue this Matter farther, I recommend Mr. Hayes's *Treatise of* Fluxions, *publish'd since the* first Edition *of the* Compendium.

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ALGE-

ALGEBRA.

INTRODUCTION.

1. THE Name *Algebra*, Dr. *Wallis* acquaints us, is derived from the two first Words of *Al-giabr*, *Wolmokabala*, which in the *Arabick* Tongue signifies, *The Art of Restitution and Comparison*, or *The Art of Resolution and Equation*. It was unquestionably known to the Ancient *Grecians* (for there are plain Footsteps of it in *Theon* upon *Euclid*, in *Pappus*, and especially in *Archimedes* and *Apollonius*) but it was studiously conceal'd by them, and kept as a great Secret. It was yet of more ancient Use among the *Arabians*, who are supposed to have received it from the *Persians*, and they from the *Indians*. From the *Arabs*, the *Moors* and *Saracens* brought it into *Spain*: From whence it came into *England*; as did also the Use of the Numeral Figures, Mathematicks in general, and Astronomy in particular, much about the

ſame Time. The firſt *European* Writer of *Algebra* was one *Lucas de Burgo*, or *Lucas Pacciolus*: His Book was Printed at *Venice* in *Italian*, A. D. 1494. a good while before we knew any thing of *Diophantus*. This *Great Art*, as *Lucas de Burgo* and *Cardan* call it, may be defined, or rather deſcribed, to be *an Analytical Way of Demonſtration, where, aſſuming the Quantity ſought as if it were known and granted; by the Help of one or more Quantities really given or known, we proceed by* Conſequences, *till at laſt the Quantity firſt ſought, and only ſuppoſed to be known, is found equal to ſome real known Quantity, and ſo is it ſelf (of Conſequence) diſcovered.*

2. The Quantity thus ſought is called the *Root*, which being unknown cannot be really expreſs'd; but may be deſign'd by any Symbol or Character at pleaſure. I (with moſt others) uſe *Vowels* for unknown, and *Conſonants* for known, or given Quantities; as (*a*) or (*e*) for a Root ſought Tho' *Des Cartes* and his Followers and moſt Foreign Writers uſe the laſt Letters of the Alphabet x, y, and z, for unknown Quantities, and the former Letters, as a, b, c, d, *&c.* for known ones.

3. The Art of *Algebra* doth much depend on the Knowledge of ſome certain Quantities, by the Ancients called *Coſſick* Quantities, but moſt uſually Powers. Which Quantities ariſe from a Rank of Numbers in continual Proportion Geometrical, beginning from Unity: For every Term, but the firſt (or Unity) is call'd ſome *Power*, as in theſe, 1, 2, 4, 8, 16, 32, *&c.* The firſt Term (2) is the

the *Root* or firſt Power; the ſecond Term (4) is the *Square* or ſecond *Power*. The third Term (8) is the *Cube* or third *Power*, &c.

So in a Rank of Fractions deſcending from Unity in the ſame Proportion: as 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, *&c.* the ſeveral Terms have alſo the ſame Names of *Powers*.

4. Hence 'tis clear, that any Number being taken as a Root, the ſecond *Power* or *Square* will be produc'd by Multiplying the Root by it ſelf; the third *Power* or *Cube*, by Multiplying the ſecond Power by the *Root*, &c.

5. If over ſuch a Rank of Numbers in Geometrical Proportion continued, there be placed a Series of Numbers beginning with (1) and proceeding orderly in an Arithmetical Progreſſion, as 1, 2, 3, 4, 5, 6, *&c.* thoſe Numbers are properly called *Indexes* or *Exponents*: Becauſe they both ſhew the Order and Place of each Power; and alſo its *Dimenſions*: (*i. e.*) how often the Root is Multiplied into it ſelf to produce that Power: For as many Units as are in the *Exponent*, ſo many Powers is that Number from Unity (*v. g.*) if the Index be 5, the Power under it, is the fifth Power, or the *Biquadrate* multiplied by the Root.

6. The Sum ariſing from the Addition of any two Indexes makes another, ſhewing what *Power* would be produced by the Multiplication of the two *Powers* Anſwering: So, that by adding them, Multiplication is made in the Numbers themſelves,

on which depends the Nature and Uſe of the Logarithms : And ſo, on the contrary, Diviſion in the correſponding Numbers anſwers to the Subtraction of the Indexes one from another.

7. Powers from any Letter repreſenting the Root are produced, by repeating it ſo often as the Index of the Powers requires ; ſo from the Root (*a*), the Square is (*aa*), the Cube (*aaa*), the Biquadrate (*aaaa*), the fifth Power (*aaaaa*), the ſixth (*aaaaaa*), the ſeventh (*aaaaaaa*), &c. or as *Des Cartes* by the Indexes chuſes rather to expreſs it, a^2, a^3, a^4, a^5, a^6, a^7, *&c.* which is ſhorter, and more convenient in many Caſes.

8. In a Rank of Fractional Numbers deſcending from Unity (as before) the Indexes are all Negative, and are imagined to have this Sign (—) before them, which is implyed in Writing them Fraction-wiſe, thus, $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, $\bar{5}$, *&c.*

9. *Like Quantities* in Algebra are ſuch as are expreſs'd by the ſame Letters equally repeated in each Quantity ; as *a* and *a*, *b* and *b*, *b c d* and *b c d*. *Unlike Quantities* are ſuch as are either expreſs'd by different Letters, or by the ſame Letters unequally repeated, as *a* and *b* : or *a* and *a a* : *b* and *b b b*, &c.

Like Signs are, when they are all of the ſame Nature, as all *Poſitive*, or all *Negative.*

Unlike Signs are, when ſome of them are *Poſitive* or *Affirmative*, and others *Negative.*

Simple

Simple Quantities are ſuch as conſiſt of but one Member, but *Compound ones* are ſuch as are compounded of 2 or more Members connected by the Signs + or —.

Co-efficients are ſuch Numbers as are prefixed before any Letters: as in 5 *a*, 7 *c*. The *Co-efficients* are 5 and 7: They are ſo called, becauſe they are ſuppoſed to help to make a Product or Rectangle, with the Quantity expreſs'd by thoſe Letters: As will be farther explained hereafter.

All Quantities expreſs'd by Letters which have no Number prefix'd before them, are ſuppoſed to have 1 for a *Co-efficient*: Becauſe every thing contains it ſelf once: Thus, *b* is the ſame as 1 *b*.

If a Letter or Quantity have not the Negative Sign — before it, 'tis always ſuppoſed to have the Affirmative one +.

ADDI-

ADDITION.

ADDITION in *Algebra* or *Species*, is performed in general, by conjoyning the Quantities propos'd, preserving their proper Signs. And the proper Mark or Sign of Addition is $+$: Which is always supposed to belong to the Quantity which follows it.

Thus if to $3a$ / you add $2a$ } the Sum is $3a + 2a$, or $5a$.
and $A + 2b$ when added to $C + bb$, makes $A + C + 2b + bb$.

Addition in Algebra may easily be learnt by observing the following particular Rules.

RULE I.

When *Simple* and *Like* Integers having *Like* Signs are to be added, collect the Numbers (or Co-efficients (all into one Sum, and to that Sum annex the Letters by which any of the Quantities was express'd, and lastly prefix the proper Sign.

Thus $-b$	and $+bcd$	and $-36de$
$-2b$	$+2bcd$	$-4de$
make $-3b$	$+4bcd$	make $-40de$
	make $7bcd$	

RULE

RULE II.

When two Simple and like Quantities have equal Numbers prefix'd, and unlike Signs, the Sum is 0.

Thus, $+3a$	and $-bb$	and $-7dce$
$-3a$	$+bb$	$+7dce$
00	00	00

N. B. The Reaſon of which is plain, if you conſider that all Quantities having negative Signs, are in Nature directly contrary to ſuch as have Affirmative ones: And therefore will always deſtroy one another. Thus, if a Man have 10 Pounds in Caſh, and run in Debt 10 *l.* that is, if to his Caſh he add a — 10 *l* (which is the proper way to expreſs a Debt) there will remain nothing: For the Debt or — 10 *l.* will deſtroy the Caſh or + 10 *l.* So alſo if a Man owe 10 *l.* and have nothing to pay it; then hath he a — 10 *l.* or is 10 *l.* worſe than nothing. And if any Perſon give him 10 *l.* or add a + 10 *l.* to his — 10 *l.* the Sum will be nothing; but however the Man will, tho' worth nothing, be 10 *l.* better than he was before.

So that 'tis a general Rule in Algebra, that to add —. is the ſame thing as to take away +, and to take away —, is the ſame thing as to add +, and to take away + is all one as to add —.

RULE III.

When two Simple and like Quantities are given, having unlike Signs, and unequal Numbers prefix'd; Subtract the lesser Quantity from the greater, and to the Remainder annex the Letters due, prefixing the Sign that belongs to the greater Quantity.

Thus, $\begin{array}{r} +3a \\ -\ a \\ \hline +2a \end{array}$ and $\begin{array}{r} -8b \\ +2b \\ \hline -6b \end{array}$

The Reason of which is clear from what was said in the *N. B.* of the last Rule.

RULE IV.

When three or more Simple and like Quantities have unlike Signs, collect the Affirmative Quantities into one Sum, and the Negative into another, then proceed as in the third Rule; and the Difference between them is the Sum sought.

Thus, $\left.\begin{array}{r} -7a \\ -3a \end{array}\right\} = -10a$

$\left.\begin{array}{r} +5a \\ +9a \end{array}\right\} = +14a$

$4a = \text{Sum}.$

RULE

RULE V.

When two or more Simple and unlike Quantities are proposed, write them down one after another, without altering their Signs.

$$\begin{array}{lr} \text{Thus,} & +3a \\ & +4b \\ \hline & +3a+4b \end{array}$$

From due Apprehension of, and mature Consideration on which Rules, the Addition of Compound Quantities may be easily performed. Thus,

$$\begin{array}{lr} & +3ee+7bb \\ & -\ \ ee-2bb \\ & +\ \ ff+3ff \\ \hline \text{Sum,} & 3ee+7bb-ee-2bb+ff+3ff. \\ \hline \text{Contracted} & +2ee+5bb+3ff. \end{array}$$

Addition of Indexes is performed after the same Manner as that of Algebraical Quantities.

Thus: To 3 add 3, the Sum is 6; where both are the Indexes of Integer Numbers: But to 3 add $\bar{2}$, the Sum will be 1; to $\bar{5}$ add 3, the Sum will be $\bar{2}$, *&c.*

SUBTRACTION.

SUBTRACTION in the General, is taking a lesser Quantity out of a greater, in order to find the Difference between them, which Difference is in Common Arithmetick, usually call'd the *Remainder*, as the lesser Quantity to be subtracted, is call'd the *Subtrahend*. The Difference is usually noted with x or d.

The general Mark or Sign of Subtraction in Algebra, is —, which whenever it comes between two Quantities, belongs to the latter of them.

Subtraction in Algebra is perform'd by conjoining the Magnitudes proposed together, but always changing the Signs of the Subtrahend.

Thus: If from $4a$ you would subtract a; by changing the Sign of the Subtrahend, it will stand thus:

$$4a - a = 3a; \quad \text{or thus,} \quad \begin{array}{r} 4a \\ -a \\ \hline x = 3a \end{array}$$

So also, If from $6b - 54bb + 4fg$, you were to subtract $6b - 54bb - 4fg$; there would remain only $8fg$; for from $6b - 54bb$
$- 4fg$

— $4fg$, taking the ſame Quantities by changing of their Signs, or adding them to it with their Signs already chang'd, and comparing and contracting them as taught in Addition, there will remain nothing but $8fg$.

For to ſubtract +, is the ſame as to add —, and to ſubtract —, is all one as to add +.

The Reaſon of which will be plain from the Inſtance given in the *N. B.* of Addition, and now applied to Subtraction. Suppoſe a Man have but 10 *l.* in Caſh: 'Tis plain, that if from him you take 10 *l.* he can have nothing left, which is Common Subtraction. Or if you make him run into Debt 10 *l.* that is, add to his real Caſh a — 10 *l.* he will ſtill be worth nothing in Reality. But if any one will pay that 10 *l.* for him, or which is all one, take away the Debt of 10 *l.* or ſubtract the — 10 *l.* he doth as much Service as if he added a real 10 *l.* to his Caſh. Wherefore all manner of Subtraction in Compound Quantities may eaſily be perform'd by only obſerving the general Rule of Subtraction, to change all the Signs of the Subtrahend; and then comparing the ſeveral Members together and contracting them.

One Inſtance is enough: Suppoſe from $36d + 5nn - 72bb$, you were to take $30d + 5nn - 72bb$, write them down one under another, changing all the Signs of the lower Rank. Thus,

$$\begin{array}{r} 36d + 5nn - 72bb \\ -30d - 5nn + 72bb \\ \hline 6d \qquad\qquad\qquad\quad \end{array}$$

 And

And then comparing them together, you will find all destroy'd, or to vanish by the Contrariety of the Signs, but 6 *d*, which therefore is the true Difference between those Quantities.

Subtraction of Indices is done as in Algebraick Quantities, by changing the Signs of the Subtrahend. Thus, If from 3, the Index of the Logarithm of an Integer Number, you take $\bar{2}$, the Index of the Logarithm of a Fraction, the Difference will be 5; if from $\bar{3}$ you take $\bar{2}$, the Remainder will be $\bar{1}$; and from $\bar{3}$ taking 2, the Difference will be $\bar{5}$.

MULTIPLICATION.

FOR *Multiplication* in Algebra, the General Rule is, to conjoyn the Quantities propos'd by the Sign of Multiplication (×.) Which Sign, when the Quantities to be multiplied are expreſs'd by but one or two Letters, is uſually omitted, and the Quantities written down like Letters in a Word, as you will find below.

The General Rule about this Sign is, that *like* Signs always give +, and *unlike* Signs, — in the Product. That is, if the Signs are either both Poſitive, or both Negative, the Product will always be Poſitive; but if one Factor be Poſitive and the other Negative, the Product is always Negative: The Reaſon of which follows below.

In Algebraick Multiplication, 'tis moſt commodious to begin to multiply at the left Hand, becauſe we write that way.

Particular Rules.

I. If Two or more ſingle Quantities expreſs'd by Letters, whether *Like* or *Unlike*, are to be multiplied into one another, and have no Numbers or Co-efficients prefix'd, joyn them together like Letters in a Word.

Thus,

Thus, a multiplied by b, makes ab; and

ab	and mno
by dc	by pqr
Product $abdc$	Product $mnopqr$

II. If two or more ſimple Quantities, whether Like or Unlike, are to be multiplied, and have Numbers or Co-efficients before them; firſt multiply the Co-efficients one into another, and then to the Product annex the Letters of both Quantities, ſo ſhall the new Quantity be the true Product.

Thus, If $3a$ were to be multiplied by $4b$, the Product will be $12ab$,

$36mn$	$15abc$
by $4b$	by $9def$
Product $144mnb$	Product $135adbecf$

III. The Multiplication of Compound Quantities depends entirely on the preceding Rules; only you muſt be ſure to multiply every Member of one Factor into every one of the other, and obſerve the Rule above given about the Signs. Thus,

$$\begin{array}{l} a+d-c \\ \text{by } g-b+f \\ \hline \text{Prod. } ga+gd-gc-ba-bd+bc+fa+fd-fc. \end{array}$$

IV. When

IV. When one and the ſame Quantity is multiplied by its ſelf, the Product is call'd the Square of that Quantity; and if that Square be multiplied again by the Root, the Product is called the Cube, &c. And this way of producing the Powers of Numbers or Quantity, is call'd by Dr. *Pell*, and ſome others, *Involution*.

Thus, $a \times$ by $a = aa$ the Square, and $aa \times$ by $a = aaa$ the Cube or third Power, and $aaa \times a = aaaa$, or a^4 the Biquadrat, &c.

'Tis the ſame thing in Compounds.

$$\begin{array}{c} a + b \\ a + b \\ \hline \end{array}$$

Product $aa + 2ab + bb =$ the Square of $a + b$, and if that Square be multiplied again by $a + b$, it produces $aaa + 3aab + 3bba + bbb$, which is the Cube of $a + b$.

N. B. That in Algebraick Multiplication, like Signs muſt give a Poſitive Product, and unlike Signs a Negative one, may be thus Demonſtrated.

I. Since Multiplication is only adding one Factor (or the Multiplicand) to its ſelf as often as there are Units in the other, or in the Multiplicator: Therefore $+$ multiplying $+$, muſt produce $+$; ſince the Sum ariſing from the Addition of Poſitive Quantities, muſt be Poſitive.

II. A

Thus, a multiplied by b, makes ab; and

ab	and mno
by dc	by pqr
Product $abdc$	Product $mnopqr$

II. If two or more simple Quantities, whether Like or Unlike, are to be multiplied, and have Numbers or Co-efficients before them; first multiply the Co-efficients one into another, and then to the Product annex the Letters of both Quantities, so shall the new Quantity be the true Product.

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$36mn$	$15abc$
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Product $144mnb$	Product $135adbecf$

III. The Multiplication of Compound Quantities depends entirely on the preceding Rules; only you must be sure to multiply every Member of one Factor into every one of the other, and observe the Rule above given about the Signs. Thus,

$$a+d-c$$
by $g-b+f$

Prod. $ga+gd-gc-ba-bd+bc+fa+fd-fc$.

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$$\begin{array}{c} a+b \\ a+b \\ \hline \end{array}$$

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I. Since Multiplication is only adding one Factor (or the Multiplicand) to its ſelf as often as there are Units in the other, or in the Multiplicator: Therefore $+$ multiplying $+$, muſt produce $+$; ſince the Sum ariſing from the Addition of Poſitive Quantities, muſt be Poſitive.

II. A Quantity with an Affirmative Sign, multiplying one that hath a Negative one, muſt produce a Negative Product; for 'tis only adding the Negative Factor to it ſelf, as often as there are Units in the other. Now never ſo many Negatives added together, will ſtill be Negative; and ſo the Product muſt have a Negative Sign.

Thus, $\left.\begin{array}{r} -6 \\ \text{by } +2 \end{array}\right\}$ gives — 12 in the Product, becauſe 'tis only taking — 6, as often as there are Units in 2, *i. e.* twice; therefore the Prod. muſt be — 12.

$$\begin{array}{r} -6 \\ +2 \\ \hline -12 \end{array}$$

III. Negative Quantities multiplying Poſitive ones, muſt produce a Negative Product; becauſe, in this Caſe the *Multiplicator*, having a Negative Sign, works on the *Multiplicand* by Subtraction; which therefore muſt be ſubtracted or made Negative (by changing its Sign) as often as there are Negative Units in the Multiplier. Thus, If + 6 be multiplied by — 2, the Product here alſo muſt be — 12; becauſe the Multiplicator 2 having a Negative Sign, ſhews that the Multiplicand muſt be ſubtracted twice from Reality, or twice repeated with a Negative Sign; wherefore the Product muſt be Negative.

$$\begin{array}{rr} & +6 \\ & -2 \\ \hline \text{Product} & -12 \end{array}$$

IV. Negatives multiplying Negatives muſt produce an Affirmative or Poſitive Product; becauſe Multiplication by a Negative Quantity being only a Subtraction, or changing the Sign of the Multiplicand as often as there are Units in the Multiplicator; and ſince Subtracting —, is the ſame as Adding +, (as was ſhewed in Subtraction) the Defect of the Multiplicand is by this means taken away, and conſequently the Product will be Affirmative.

Thus, If — 6 be multiplied by — 2, the Product will be + 12; becauſe the Multiplicator 2 having here a Negative Sign, acts on the Multiplicand by Subtraction, ſubtracting its defective Sign, or changing it into an Affirmative one, (for to ſubtract —, is to add +) as often as there are Units in its ſelf. Wherefore the Product muſt have a Poſitive Sign. *Q. E. D.*

II. A Quantity with an Affirmative Sign, multiplying one that hath a Negative one, muſt produce a Negative Product; for 'tis only adding the Negative Factor to it ſelf, as often as there are Units in the other. Now never ſo many Negatives added together, will ſtill be Negative; and ſo the Product muſt have a Negative Sign.

Thus, $\left.\begin{array}{r} -6 \\ \text{by } +2 \end{array}\right\}$ gives -12 in the Product, becauſe 'tis only taking -6, as often as there are Units in 2, *i. e.* twice; therefore the Prod. muſt be -12.

$$\begin{array}{r} -6 \\ +2 \\ \hline -12 \end{array}$$

III. Negative Quantities multiplying Poſitive ones, muſt produce a Negative Product; becauſe, in this Caſe the *Multiplicator*, having a Negative Sign, works on the *Multiplicand* by Subtraction; which therefore muſt be ſubtracted or made Negative (by changing its Sign) as often as there are Negative Units in the Multiplier. Thus, If $+6$ be multiplied by -2, the Product here alſo muſt be -12; becauſe the Multiplicator 2 having a Negative Sign, ſhews that the Multiplicand muſt be ſubtracted twice from Reality, or twice repeated with a Negative Sign; wherefore the Product muſt be Negative.

$$\begin{array}{r} +6 \\ -2 \\ \hline \text{Product } -12 \end{array}$$

IV. Nega-

IV. Negatives multiplying Negatives muſt produce an Affirmative or Poſitive Product; becauſe Multiplication by a Negative Quantity being only a Subtraction, or changing the Sign of the Multiplicand as often as there are Units in the Multiplicator; and ſince Subtracting —, is the ſame as Adding +, (as was ſhewed in Subtraction) the Defect of the Multiplicand is by this means taken away, and conſequently the Product will be Affirmative.

Thus, If — 6 be multiplied by — 2, the Product will be + 12; becauſe the Multiplicator 2 having here a Negative Sign, acts on the Multiplicand by Subtraction, ſubtracting its defective Sign, or changing it into an Affirmative one, (for to ſubtract —, is to add +) as often as there are Units in its ſelf. Wherefore the Product muſt have a Poſitive Sign. *Q. E. D.*

DIVISION.

DIVISION in Algebra in the General, is reducing the Dividend and Divisor to the Form of a Fraction, which Fraction is the Quotient. Thus, if $a\,b$ were to be Divided by $c\,d$: It is in Algebra placed thus $\frac{a\,b}{c\,d}$ and this Fraction is the Quotient.

Some express Division thus $c\,d)\ a\,b$ or $a\,b \div c\,d$, which Character $\div$ is the ordinary Sign of Division.

For duly performing the Work of Algebraick Division, observe these Rules.

I. When the Dividend is equal to, or the same with the Divisor, the Quotient is 1. (not 0.) For every thing is it self once. Therefore when ever this happens, as it will often do in Equations, remember always to place 1. in the Quotient.

II. When the Quotient is express'd Fraction-wise, (as in Simple Division) if the same Letters are found equally repeated both above and below the Line of Separation, you may cast off those equal Letters, and the Remainder will be the true Quotient. Thus,

$$\frac{ab}{db} = \frac{a}{d} \qquad \text{and} \ \frac{abc}{ab} = c, \text{ \&c.}$$

The Reaſon of which is plain, becauſe the Fraction $\frac{a}{d}$ being Multiplied both above and below the Line by b (in the firſt Inſtance) hath not its Value at all alter'd thereby, and therefore when the Quantity $\frac{ab}{db}$ comes to be Divided by the common Multiplier b, which is done by caſting off b, the ſame Value will remain, and $\frac{a}{d}$ is the Quotient.

III. When the Quantities expreſs'd by Letters have any Co-efficients, divide them as in Common Arithmetick, and to the Quotients annex the Quantities expreſs'd by the Letters. Thus,

$$\frac{360\,ab}{24\,b} = 15\,a.$$

Only remember that if the Quantities have unlike Signs, the Quotient muſt have a Negative Sign, if like Signs a Poſitive one.

IV. The General way of Diviſion of Compound Quantities is like the ordinary way in Common Arithmetick; reſpect being had to the Rules of

 Alge-

Algebraic *Addition*, *Subtraction* and *Multiplication:* Obſerving always this, That like Signs give $+$, and unlike $-$ in the Quotient, as was ſaid before.

You muſt always take care to divide every Member, or part of the Dividend, by its *proper Diviſor:* (*i. e.* by ſuch an one whoſe Letters ſhew it to be of the ſame kind with the other): For you muſt always place ſuch a Letter in the Quotient, as will, when Multiplied into the Diviſor, produce the Dividend, (or at leaſt a good part of it) ſince the Dividend is a Rectangle under the Diviſor and Quotient. Thus,

$$a+b)\ aa+ab-ca-cb\ (a-c$$
$$aa+ab$$
$$0 \qquad -ca-cb$$
$$-ca-cb$$
$$0$$

Another Example.

$$zz-16)\ z^6-8z^4-124zz-64\ (z^4+8zz+4$$
$$z^6-16z^4$$
$$8z^4-124zz$$
$$8z^4-128zz$$
$$4zz-64$$
$$4zz-64$$
$$0$$

Another

Another Example.

$$3dd-ee)\,9d^4+12d^3e-4deee-eeee\,(3dd+4de+ee$$
$$9d^4-3ddee$$
$$12d^3e+3ddee-4dee$$
$$12d^3e\quad -4dee$$
$$3ddee-e^4$$
$$3ddee-e^4$$
$$0$$

N. B. That the ſame Reaſon for like Signs giving a Poſitive Quotient, and unlike a Negative one, holds here as well as for the Product in Multiplication, is clear, from conſidering the Nature of Diviſion.

For every Dividend being nothing but a Product made by multiplying the Quotient by the Diviſor; the Sign of each Factor muſt be ſuch, as according to the former Rules in Multiplication, can produce the Dividend. Wherefore if the Dividend be Poſitive, and divided by a Negative, the Quotient muſt be Negative: Since if it be Poſitive, it cannot produce the Dividend by Multiplication into the Diviſor. If the Dividend be Negative, either the Diviſor or Quotient muſt have a Negative Sign; but they cannot be both Negative. For then they would produce a Poſitive Dividend.

Wherefore 'tis plain, that if the Dividend be divided by a Quantity, which hath a *like* Sign with it, the Quotient muſt be *Poſitive*; but if by one having an *unlike* Sign, the Quotient will be *Negative*.

O F

OF

FRACTIONS.

A *Fraction* is a broken Number or Quantity, expressing the Parts of some Integer. It consists of two Parts with a Line of Separation placed between them: Of which, that above the Line is call'd the *Numerator*, because it *Enumerates*, or tells you how many of the Parts of the Integer the Fraction contains: And that below the Line is call'd the *Denominator*; because it *Denominates*, or Expresses the Nature of the Parts the Integer is supposed to be divided into. Thus,

Suppose $a=3$ and $b=4$, then will $\frac{a}{b}$, or $\frac{3}{4}$ be a Fraction, expressing, that some Integer being divided into 4 Parts or *Quarters*, there is taken 3 of them, or 3 Quarters.

A Fraction is either *Proper*, when the *Numerator* is less than the *Denominator*; as $\frac{3}{4}$: Or *Improper*, when the *Numerator* is equal to it, or greater: As $\frac{4}{4}$ or $\frac{4}{3}$ are improper Fractions; be-

cause one expresses the whole *Integer*, and the other more than the *Integer*; however 'tis often of good use to express Quantities after this way.

The Operations about Algebraic Fractions, or Fractions express'd by Letters, are much of the same Nature with those in Common Arithmetick.

I. All Fractions ought first to be reduc'd to their lowest Terms; which is done by dividing both Numerator and Denominator, by their *greatest Common Divisor*; that is, the *greatest* Quantity which can divide both. For then the Quotient will be a Fraction of the same Value as the former, but in the smallest Terms that can be. Thus, $\frac{3aa}{6a}$ by dividing both Parts by $3a$, will be brought down to $\frac{a}{2}$ or $\frac{1}{2}a$ and $\frac{4a^6}{6a^4}$ being divided by its greatest Common Divisor $2a^4$, will be reduced to $\frac{2aa}{\quad}$.

$$\begin{array}{ll} 4z)\ 36zz & \\ \hline 4z)\ 4bz + 16dz & \left(\dfrac{9z}{b+4d}\right. \end{array}$$

And this may most times be done by Inspection, by casting out of both *Numerator* and *Denominator*, such Letters as are multiplied into both of them, as in these Examples.

But

But ſuch greateſt Common Diviſor may be found in all Caſes, where the Eye cannot readily diſcover it, by dividing the Denominator by the Numerator, and the laſt Diviſor by the Remainder, if any be; and ſo on, until there come to remain nothing: And then that laſt Diviſor is the greateſt Common Meaſure. But if Unity, or 1 remain at laſt, then the Fraction was in its loweſt Terms at firſt, and cannot be reduced to any ſmaller Terms. This Practice is the ſame as in Vulgar Fractions; and you have an Example of it in *Species* in *Ward*'s Algebra, *Chap.* 4.

II. To reduce any Integer, as b or $a + c$ to the Form of an improper Fraction, draw the Line of Separation, and under it write 1, then it will ſtand $\frac{b}{1}$ or $\frac{a+c}{1}$, which, tho' in the Form of Fractions, are not altered, becauſe 1 neither multiplies nor divides.

If a *Denominator*, as d were given: Firſt multiply the given Integer by ſuch Denominator, and then write the Denominator under the Product. Thus,

$$\frac{db}{d} = b, \text{ and } \frac{da + dc}{d} = a + c.$$

III. To reduce Fractions of different Denominators, to others of the ſame Value, that ſhall have a Common Denominator; (*which Operation muſt always precede Addition and Subtraction in Fractions.*) You muſt firſt bring the Fractions down as low as you can; (by *Rule* 1.) then multiply a-croſs the

the Numerator of the firſt, into the Denominator of the ſecond, for a new Numerator for the firſt Fraction; then the Numerator of the ſecond into the Denominator of the firſt, for a new Numerator for the ſecond Fraction; and laſtly, multiply the Denominators one into another, for a Common Denominator. Thus,

Let $\frac{a+b}{b}$ and $\frac{bb}{f}$ be given; and they will by this Rule be reduc'd to $\frac{fa+fb}{df}$, and $\frac{dbb}{df}$: Fractions in Value equal to the former.

The Reaſon of which is plain, for each Fraction is multiplied and divided by the ſame Quantity or Letter, and therefore muſt retain the ſame Value as before, tho' reduc'd to another Form:

$$\frac{4}{6} \qquad \frac{3}{4}$$

$$\frac{16}{24} \qquad \frac{18}{24}$$

For every Fraction being multiplied by multiplying its Numerator, but divided by dividing it; and being alſo multiplied by dividing the Denominator, and divided by multiplying it.

It follows, That each Fraction will gain as much by the Multiplication of its Numerator, as it loſes by the Multiplication of its Denominator: And *Vice verſâ*, in caſe of Diviſion by one and the ſame Quantity.

If there are more than two Fractions, every Numerator muſt be multiplied continually into all the Denominators but its own; and the Denominators one into another continually for a new Denominator. *Ex. gr.* $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}$, will be reduced to this Form $\frac{ayz}{xyz}, \frac{bzx}{yzx}, \frac{cyx}{zyx}$, which are Fractions of the ſame Value as the former (as is apparent by ejecting the Common Letters) but reduced to a Common Denominator.

IV. And when this is once underſtood, *Addition* and *Subtraction* in *Fractions* are performed by only Adding or Subtracting the *Numerators*, and Subſcribing the *Common Denominators* before found. Thus,

If the Fractions $\frac{a+b}{d}, \frac{hh}{f}$ were to be *Added* or *Subtracted*; they will ſtand, when reduced (by *Rule* 3.) in this form, $\frac{fa+fb+dhh}{df}$ or $\frac{fa+fb-dhh}{df}$: The former of which, is the *Sum*, the latter the *Difference* of the two given *Fractions*.

V. *Multiplication in Fractions*, is perform'd by multiplying the Numerators into one another, for a new Numerator, and the Denominators for a new Deno-

Denominator, the Fractions having been first reduced to their lowest Terms. Thus,

$$\frac{a}{b} \times \frac{d}{c} = \frac{da}{bc} \text{ and } \frac{a+b}{c} \times \frac{a-b}{d} = \frac{aa-bb}{cd}$$

Hence, If any Fraction be multiplied by the Denominator, or by some Integer, the same with it, the Numerator is the Product. As $\frac{aa}{b} \times b = aa$, for $\frac{aa}{b} \times \frac{b}{1} = \frac{aab}{b}$; which, casting off the Common Letters in both Parts, leaves aa.

Also if any Fraction be to be multiplied by some Letter or Letters that are found in every Member of the Denominator; the Multiplication may be made only by ejecting such Letters out of the Denominator: As $\frac{ab}{cd}$ multiplied by $d = \frac{ab}{c}$.

VI. *Division in Fractions*, is perform'd (after Reduction according to *Rule* 3.) by multiplying the Numerator of the Dividend by the Denominator of the Divisor, for a *Numerator*; and the Denominator of the Dividend by the Numerator of the Divisor, for a new *Denominator*. As in *Vulgar Fractions*. Thus,

$$\frac{a}{b}\Big)\ \frac{d}{c}\ \Big(\frac{bd}{ac}$$

The Reaſon of which is plain, from what was ſaid above, That a Fraction is divided by multiplying its Denominator. Thus,

$$\frac{3}{4}\Big)\ \frac{12}{16}\ \Big(\frac{48}{48}=)$$

For to divide $\frac{12}{16}$ by $\frac{3}{4}$, is to ſeek how often 3, the Numerator of the Diviſor, is in $\frac{12}{16}$, which is done by multiplying 16 by 3, and the Anſwer is $\frac{12}{48}$: But then again, becauſe $\frac{3}{4}$ is but $\frac{1}{4}$ of 3, it will be contained in $\frac{12}{16}$ 4 times oftener than 3 is; and therefore in order to bring it to a *Par*, divide the Value of the Fraction by multiplying its Denominator by 12, and the Product 48 will be the Numerator of the Quotient.

But if it happen that the Fractions have a Common Denominator, then caſt off that, and divide one Numerator by the other. Thus,

$$\frac{a}{b}\Big)\ \frac{c}{b}\ \Big(=\frac{c}{a} \quad \text{and}\ \frac{b}{a}\Big)\ \frac{bb}{a}\ \Big(=b$$

For Fractions having a Common Denominator are as their Numerators.

VII. A Mixt Quantity or Number, is that which is Part Integer, and Part Fraction. As $aa+\frac{b}{c}$: Such Quantities are reduced to the form of Improper Fractions, by firſt multiplying the Integral Part by the Denominator of the Fractional Part, then adding the Numerator to it, and

and ſubſcribing the Denominator under all. Thus the former Quantity $aa + \frac{b}{c}$ is reduced to this improper Fraction $\frac{caa + b}{c}$.

Every improper Fraction is reduced back again into its equivalent mixt Number, or Integer, by dividing the Numerator by the Denominator. Thus, $\frac{caa + b}{c}$ divided by c, quotes $aa + \frac{b}{c}$; and $\frac{aa}{1}$ divided by 1, makes aa.

OF

OF

EQUATIONS.

AN *Equation* in *Algebra*, is the mutual comparing of two equal Things of different Names or Denominations: As, ſuppoſe 3 Pounds equal to 60 Shillings = 720 Pence, which is equal to 2880 Farthings, *&c.* it may be written thus, 3 *l.* = 60 *s.* = 720 *d.* = 2880 *f.*

The *Terms of an Equation*, are the ſeveral Quantities or Parts of which every Equation is compoſed, connected together by the Signs + and —. As in this Equation $a = b + c$. The Terms are *a*, *b* and *c*; where 'tis ſuppos'd that ſome Quantity repreſented by *a*, is equal to the Sum of *b* and *c*; or to *b* and *c* added together.

Whenever a Queſtion or Problem is propoſed in *Algebra*, we always ſuppoſe the thing *ſought* or *required* to be *known* or *done*.

And then by putting the Letter *a*, or ſome other *Vowel* (many uſe the laſt Letters of the Alphabet, *x*, *y*, *z*) for the unknown Quantity, or for the thing ſought; and *Conſonants* for whatever is *known* or *given*, in order to diſtinguiſh one from the other: The Queſtion or Problem is firſt *throughly conſider'd*, and then duly *ſtated*, and after this Judiciouſly

ously compared, transformed, and varied by Addition, Subtraction, Multiplication, Division, Extraction of Roots, *&c.* according as the Nature of the Thing, and the Rules of Art direct; till at last the Quantity *sought*, or at least some Power of it, becomes equal to some *known* or *given* Quantity, and so is it self of consequence discovered.

After a Question is duly stated, 'tis proper to consider whether it be subject to any *Limitations*, or not. To which End the Writers of *Algebra* give these General Rules.

I. If the Quantities sought or required, are more than the Number of the given Equations, the Question is capable of innumerable Answers. See *Kersey*'s *Algebra*, P. 301. Vol. 1.

II. But if the given Equations, independent one upon another, are just as many as the Quantities sought: Then the Question hath only one certain and determinate Number of Answers.

If the Quantities sought or required are less in Number than the given Equations, the Question is yet more limited; and sometimes it is discoverable, that 'tis not to be resolved, by reason of such Equations being inconsistent with each other.

Equations, in order to be Resolved, must first be *Prepared* and *Reduced*; which is usually done by the following Rules, or such like.

I. If the Quantity sought, or any *Part* or *Degree* of it be in Fractions, let all be reduc'd to one Common Denomination; and then omitting the Denominators, let the Equation be continued in the

the Numerators only; or in Practice, multiply the whole by the Denominator of the Fractional part.

$$V.gr.\ \frac{a+b}{c}+d=100=B.$$

$$\text{Then first, } \frac{a+b+cd}{c}=B.$$

$$\text{And then, } a+b+cd=cB.$$

$$\text{Or if, } a-b=\frac{aa+cc}{d}+b+b.$$

Multiply all by d, and it will stand thus,

$$ad-db=aa+cc+db+db.$$

$$\text{Or if, } a-75=\frac{3}{4}bb+c-g.$$

Multiply all by 4, and

$$4a-300=3bb+4c-4g.$$

And this is call'd by *Vieta*, *Isomeria*, and by others *Conversion*.

II. When there is an Intermixture of Quantities, known and unknown in any Equation; let all the unknown Quantities (by *Transposition*) be made

made to possess only one side of the Equation, and all the known ones another. *Transposition* is always done by putting the Quantity over to the other Side with a contrary Sign to what it had before. The Demonstration of which *Rule* depends on that *Axiom*, That if to or from Equals, you add or subtract Equals, the Sums or Remainders will be equal.

Thus suppose, $a - 34 = 60$.

Then, $a = 34 + 60 = 94$.

Or if $a + b - d = b + c + e$.

Then, $a - e = c + d$.

If, $4a - 300 = 3bb + 4c - 4g$.

Then, $4a = 300 + 3bb + 4c - 4g$.

If, $36 + 44 = a - 60$.

Then, $a = 140$.

III. If the highest Power or Species of the unknown Quantity, be multiplied into any known Quantity or Quantities, let the whole be divided by such known Quantity or Quantities.

Thus if, $5aa = 30000.\ aa = 6000$.

If, $ba + ad = 1000$.

Then, $a = \frac{1000}{b+d}$.

If, $dee + dde = z$.

Then, $ee + de = \frac{z}{d}$.

And this Operation is called by *Vieta*, *Parabo-lismus*, by others *Depression*.

IV. If all the known Quantities happen to be multiplied into any Degree of the unknown one; let all be brought down (by *Division*) to the lowest degree thereof that can be.

As if, $aaaa + baaa = zzaa$.

Then by Division of all, by aa.

$aa + ba = zz$.

If $aa + ab - ac = ad - fa$.

Then will $a + b - c = d - f$. By *Division*;

And, $a = d - b + c - f$. By *Transposition*.

If $ee + 9e - 7e = 15e + 34e - 10e$.

Then will $e + 9 - 7 = 15 + 34 - 10$, by *Division*; and $e + 2 = 39$.

And farther by *Transposition*, $e = 37$.

And this is that *Rule* which *Vieta* calls *Hypo-bibasmus*.

V. If

V. If any one Member of the *Equation* be a *Surd Root*, all must be raised up to that Power, and then the *Equation continued*.

If $\sqrt{ba} + b = c$.

Then by *Transposition*,

$\sqrt{ab} = c - b$.

And by this Rule,

$cc - 2bc + bb = ab$.

And by Rule the third,

$$a = \frac{cc - 2bc + bb}{b}.$$

NOTE.

To raise up any Quantity to the Power of another, is to multiply it into it self, or to Involve *it according as the Index of that Power directs. Thus because* $\sqrt{:ab}$ *signifies the Square Root of* a b, *therefore* a b, *without the Radical Sign, will be a Square; and consequently to continue the Equality,* c — b *must be Squar'd too.*

RULE VI.

Since whenever 4 Quantities are *discretely*, or 3 *continually Proportional*, the Rectangle or Product of the two mean Terms (or the Square of the

 middle

middle one, when there are but 3) is equal to the Rectangle or Product of the Extreams (by the 12th of the 6th of *Euclid's Elements.*) Therefore 'tis very easie, and often very useful to resolve *Equations* into *Analogies*, or *Proportionals*, and *vice versâ*; which, when well understood, opens the way for the *Geometrical Construction*, and consequently for one very good way of Resolution of *Equations*.

Wherefore, Supposing the Reader tolerably versed in *Common Geometry*, as he ought to be, before he begin *Algebra*, and that he knows how to find a *Third*, a *Mean*, or a *Fourth Proportional*, *Geometrically*. I shall next shew the Construction of all kinds of *Simple Equations*, before I proceed to resolve any *Questions* or *Problems*.

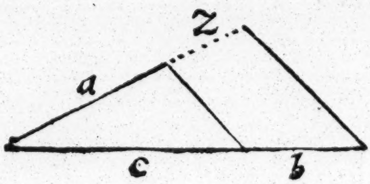
CONSTRUCTION OF EQUATIONS,

IN Algebra, *is the contriving ſuch Lines and Figures as ſhall demonſtrate the Equation, Canon, or Theorem, to be true Geometrically.*

The Method of effecting which, will be ſufficiently plain from the following Examples, and thence eaſier learn'd than by long Directions in Words.

Conſtruction of Simple Equations.

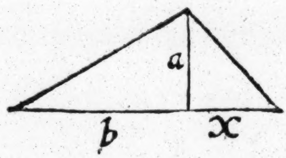
I. IF $\frac{ab}{c} = z$, then $c.\ b :: a.\ z$. 12. *e.* 6. *Eucl.*



Or

Or if, $\frac{aa}{b} = x$. Then, $b : a :: a : x$.

By 8 *e.* 6 *Euclid.*

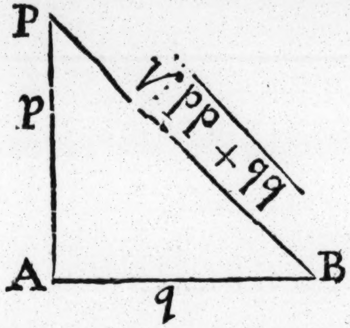


Or if, $\frac{ab+ag}{b+1} = x$. Then, $b+1 : b+g :: a : x$.

Or if it were, $\frac{ab+ag}{b-1} = x$. Then, $b-1 : b+g :: a : x$.

II. If $\frac{ab+mn}{r+s} = x$. The Construction and Solution will be more difficult; because no Letter in the Numerator is taken twice: But that it may be so, and that (a) for Instance, may be twice used, make as $a : n :: m :$ a fourth Proportional, which let be p. Then $nm = ap$, and consequently $\frac{ab+ap}{r+s} = x$. Wherefore, as by *Rule* 1. $r+s : b+p :: a : x$.

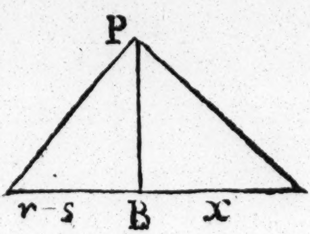
If this Equation were proposed $\frac{ab+mn}{r-s}=x$.



You may find first a middle Proportional between a and b, which suppose to be p. Also another mean Proportional between m and n, which let be q. Then will the Equation stand thus, $\frac{pp+qq}{r-s}=x$.

Right

Let therefore a [redacted]-angled Triangle be made, wherein the Perpendicular $AP=p$, and the Base $AB=q$. Therefore shall $PBq=pp+qq$, which, since according to the Equation it is to be divided by $r-s$: Make, as $r-s$ to BP $(=\sqrt{:pp+qq})$ :: so PB to a third Proportional; which shall be x sought.

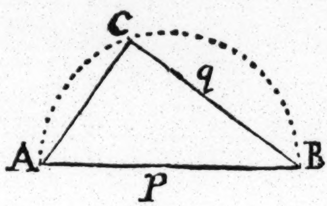


III. In

III. In this Equation $\frac{ab - mn}{c+d} = y$.

First make, as $a : m :: n :$ (4th Proportional) which let be p. Then will $\frac{ab - ap}{c+d} = y$. And consequently, (as in *Case* 1.) $c + d : b - p :: a : y$.

Or you might (as in *Case* 2.) have found a mean Proportional between a and b, as also be-



tween m and n, which being called (as there) p and q. The Equation would have stood thus, $\frac{pp - qq}{c+d} = y$. Then having taken A B $= p$, and on it as a Diameter drawn a Semicircle; and applying in BC $= q$. The □ of A C $= pp - qq$. And consequently AC $= \sqrt{: pp - qq}$; which since it is to be divided by $c+d$, make as $c+d$: A C :: A C : y, the Quantity sought.

IV. Let

IV. Let this Equation $\frac{aabc}{ffg} = z$, be proposed.

First, Find out (p) a 3d Proportional to f and a. Then fp being $= aa$. The Equation will stand thus, $\frac{fpbc}{ffg} = z$, (i. e.) $\frac{pbc}{fg} = z$.

Secondly, Find a 4th Proportional (q) to f, p, and b, saying, as $f : p :: b : q$. Then will $fq = pb$, and consequently the Equation will stand thus, $\frac{fq}{fg} = z$, (i. e.) $\frac{qc}{g} = z$. And therefore, (as by *Numb.* 1.) $g : q :: c : z$ sought.

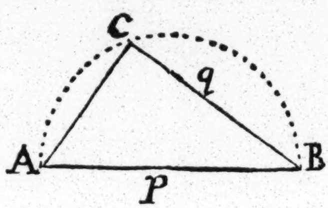
V. If this Equation $\frac{bkk}{mm} = x$ were proposed.

First, Find a 4th Proportional to m, b and k, which let be p, therefore $pm = bk$, and consequently the Equation will stand thus, $\frac{pmk}{mm} = x = \frac{pk}{m}$: Therefore $m : p :: k : x$ sought.

III. In this Equation $\frac{ab - mn}{c+d} = y$.

First make, as $a : m :: n :$ (4th Proportional) which let be p. Then will $\frac{ab - ap}{c+d} = y$. And consequently, (as in *Case* 1.) $c+d : b-p :: a : y$.

Or you might (as in *Case* 2.) have found a mean Proportional between a and b, as also be-



tween m and n, which being called (as there) p and q. The Equation would have stood thus, $\frac{pp - qq}{c+d} = y$. Then having taken AB $= p$, and on it as a Diameter drawn a Semicircle; and applying in BC $= q$. The □ of AC $= pp - qq$. And consequently AC $= \sqrt{:pp - qq}$; which since it is to be divided by $c+d$, make as $c+d :$ AC :: AC : y, the Quantity sought.

IV. Let

IV. Let this Equation $\frac{aabc}{ffg} = z$, be proposed.

First, Find out (p) a 3d Proportional to f and a. Then fp being $= aa$. The Equation will stand thus, $\frac{fpbc}{ffg} = z$, (i. e.) $\frac{pbc}{fg} = z$.

Secondly, Find a 4th Proportional (q) to f, p, and b, saying, as $f : p :: b : q$. Then will $fq = pb$, and consequently the Equation will stand thus, $\frac{fqc}{fg} = z$, (i. e.) $\frac{qc}{g} = z$. And therefore, (as by *Numb.* 1.) $g : q :: c : z$ sought.

V. If this Equation $\frac{hkk}{mm} = x$ were proposed.

First, Find a 4th Proportional to m, h and k, which let be p, therefore $pm = hk$, and consequently the Equation will stand thus, $\frac{pmk}{mm} = x = \frac{pk}{m}$: Therefore $m : p :: k : x$ sought.

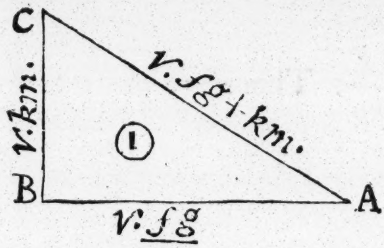
Construction of Simple Quadratick Equations.

N. B. *Simple Quadraticks, are such as have on one side of the Equation only some Power of the Quantity sought, without any other Quantity or Letter mixt with it.*

I. IF an Equation be in any of these following, or the like Forms.

$yy = ab$	therefore	$y = \sqrt{ab}$	that is $y =$ to a mean proportional between	a & b
$yy = 1c$ or c		$y = \sqrt{c}$		1 & c
$yy = \frac{3}{4}dd$		$y = \sqrt{\frac{3}{4}dd}$		$\frac{3}{4}d$ & d

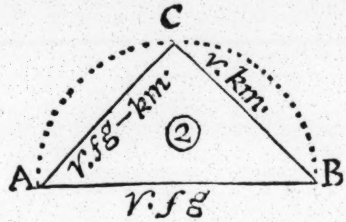
II. If this Quadratick be proposed; $yy = fg + km$. Then will $y = \sqrt{:fg + km}$.



1. When the Sign is $+$, make a Rectangled Triangle ABC, and let the Side AB $=$ to a middle Proportional between f and g, and the Side CB $=$ to a mean Proportional between k and m.

2. But

2. But if the Sign be —, make the Rectangled Triangle (*N.* 2. whose Hypothenuse A B let be =



to the mean Proportional between f and g, and the Side B C = to a mean Proportional between k and m, and the thing is plain.

III. If such an Equation as this be proposed, *viz.* $\frac{fhhdd}{fcc} = xx$. Then will it be $\frac{hhdd}{cc} = xx$, and by extracting the Square Root every where, (because all the Quantities are simple and perfect Squares) it will be $\frac{hd}{c} = x$. Therefore $c : h :: d : x$. As in *Case* 1. of *Simple Equations.*

IV. If this Equation be propos'd, $\frac{fghk}{dm} = xx$,

Find first a fourth Proportional to d, f and g, which let be p. Then $dp = fg$, and consequently $\frac{dphk}{dm} = xx$. (*i. e.*) $\frac{phk}{m} = xx$.

Again, find a fourth Proportional to m, p and h, which let be q. Then $qm = ph$, and therefore

$\frac{qmk}{m} = xx$, (i. e.) $qk = xx$. Which brings it to *Case* I. of *Simple Quadraticks*; which see, *&c.*

V. If this Equation were proposed;

$$\frac{qbcc + bccd + qbcd + bcdd}{fg + km} = xx.$$

First, reduce the Rectangles fg and km to the Squares bb and qq by finding b and q mean Proportionals between f and g, and k and m, and make the Square $nn = bb + qq$.

Then will $\frac{qbcc + bccd + qbcd + bcdd}{nn} = xx$.

Also, Because cc and cd are multiplied into $qb + bd$, find a Square $=$ to the two Rectangles $qb + bd$, and let it be pp. Therefore $\frac{ppcc + ppcd}{nn} = xx$.

Again, because pp is multiplied both into cc and cd, find another Square equal to $cc + cd$, which let be qq. Then will $\frac{ppqq}{nn} = xx$ (*i. e.*) $\frac{pq}{n} = x$. Therefore $n : p :: q : x$ sought.

Questions

Questions producing Simple Equations.

QUestion I. 200 l. *is to be divided between two Men, that one is to have* 73 l. *more than the other: What is the Share of each?*

For the whole Money put $s = 200\ l.$
For the Difference between their Shares, put $d = 73\ l.$
And let the Shares sought be a and e.

	1	$a + e = s$	by the Question.
	2	$a - e = d$	
1 + 2	3	$2a = s + d.$	

N. B. *'Tis an excellent way to keep a Register in the Margin of all the Steps in the Resolution of any Equation (according to Dr.* Pell's *Method;) for by that means it will readily appear how every Step is produced: As here the Figures* 1 + 2 *right against the third Step shew you that it is produced by adding the first and second Steps together, which is done to destroy* e, *that* a *may stand alone.*

3 ÷ 2	4	$a = \frac{s + d}{2}$:	By dividing the last Step by 2.

And thus is the Value of a presently known, (*viz.* $= 136\ l.\ 10\ s.$) and this *Theorem* also gain'd, *That half the Sum added to half the Difference of any*

ny two Quantities, is always equal to the greater of them: Which is the Senſe, in Words at length, of

$$\frac{s+d}{2} \text{ (or } \tfrac{1}{2}s + \tfrac{1}{2}d\text{)} = a.$$

And as this way you found the Value of a by Addition of the firſt and ſecond Steps; ſo you may find e by Subtraction of the ſecond Step from the firſt. Thus,

$1 - 2$	5	$2e = s - d$: Wherefore,
$5 \div 2$	6	$e = \dfrac{s-d}{2}$

That is in Words, *Half the Sum of any two Quantities leſs half their Difference is always equal to the leſſer of them:* Which is a Canon or Theorem that will find e to be 63 *l.* 10 *s.*

So that having the Sum and Difference of any two Quantities or Numbers, 'tis, you ſee, very eaſie to diſcover the Numbers themſelves.

Queſtion II. 200 l. *is ſo diſtributed between two Men, that if the Share of one be divided by that of the other, the Quotient will be* 3. *How much had each?*

For the Shares, put a and e;
Then $a + e = s = 200$.

$$\frac{a}{e} = q = 3.$$

	1	$e = s - a$
	2	And since $\frac{a}{e} = q$. If you multiply both by e: $a = eq$.
$2 \div q$	3	$\frac{a}{q} = e$, by which means e vanishes, being expressed another way.
1, 3.	4	$s - a = \frac{a}{q}$ by considering the first and second Steps; (which is always implyed by the Comma's in the Margin.)
$4 \times q$	5	$sq - qa = a$, Multiplying both by q.
Transp.	6	$sq = a + qa$, by Transferring qa.
$6 \div 1 + q$	7	$\frac{sq}{1+q} = a$, by Dividing each part by the Co-efficient $1 + q$; that is, $a = 150\,l$.
	8	$e = 50\,l.$ because a was 3 times as much as e by the Question.

If instead of the Sum of the 2 Shares the Difference $d = 100$ had been given, as also the Quotient $\frac{a}{e} = 3 = q$; and a and e required?

Then,

Then,

	1	$e + d = a.$
	2	$a = eq$ $\Big\{$ by the second Step of the former Question.
1, 2,	3	$e + d = eq.$
Transp.	4	$d = eq - e.$
$4 \div q - 1$	5	$\dfrac{d}{q - 1} = e = 50\,l.$ Wherefore
		$a = 3e = 150\,l. = d + e.$

Question III. *Two Men have between them* 56 l. *and the greater Share is to the lesser as* 5 *to* 2, or *as* r *to* t. *What had each?*

Put a and e for the 2 Shares, and $s = 56\,l.$

	1	$s = a + e.$
	2	$r : t :: a : \dfrac{ta}{r} = e.$

This Step arises only by saying according to the *Golden Rule*; if r give t, : What shall a give?

The fourth Term is $\dfrac{ta}{r} = e$, which finds a new Notation for e.

1, 2,	3	$a + \frac{ta}{r} = S.$
$3 \times r$	4	$ra + ta = rS.$
$4 \div r + s$	5	$a = \frac{rs}{r+t} = \frac{280}{7} = 40\,l.$

And consequently,

1, 5,	6	$e = \frac{80}{5} = 16\,l.$

If instead of the Sum, the Difference $d = 24$ had been given. Then,

	1	$a - \frac{ta}{r} = d$ $\Big\{$ by working for e as in Step 2 of the last.
$1 \times r$	2	$ra - ta = rd.$
$2 \div r - t$	3	$a = \frac{rd}{r-t} = \frac{120}{3} = 40.$
	4	$e = a - d = 40 - 24 = 16$, as before

Question IV. *One having a certain Number* o *Eggs, left (without breaking any) half that Num ber and half an Egg at one Place: Half the* R*e mainder, and half an Egg at the second Plac Half the Remainder and half an Egg at thir Place: And then he had one Egg left: H*o *many had he at first?*

For the Number of Eggs put a.
And for one Egg put $1 = b$.

1 | $\frac{a}{2} + \frac{b}{2} =$ { What he left at th first Place. And,

2 | $\frac{a}{2} - \frac{b}{2}$ or $\frac{a-b}{2} =$ { To the fir Remaind

And if to avoid the trouble of *Vulgar*, you wou exprefs it by *Decimal Fractions*: Then the fi

Eggs left will be $\frac{a}{2} + :5$, and the first R

mainder, $\frac{a}{2} - :5$.

3 | $\frac{a}{4} - \frac{b}{4} + \frac{b}{2}$: or $\frac{a}{4} - :25 +$
$=$ to what he left at the secon Place: And,

4 | $\frac{a}{4} - \frac{b}{4} - \frac{b}{2}$: or $\frac{a}{4} - :25 -$
will be the second Remaind

	5	$\frac{a}{8} - \frac{b}{8} - \frac{b}{4} + \frac{b}{2}$, or $\frac{a}{8} -$: 125 — : 25 + : 5 = to what he left at the third Place.
	6	$\frac{a}{8} - \frac{b}{8} - \frac{b}{4} - \frac{b}{2}$, or $\frac{a}{8} -$: 125 : — : 25 — : 5 =, the third Remainder; which by the Question is = to b or 1. Therefore,
	7	$\frac{a}{8} - \frac{b}{8} - \frac{b}{4} - \frac{b}{2} = b$, or $\frac{a}{8} -$: 125 — : 25 — : 5 = 1. Wherefore,
Transp.	8	$\frac{a}{8} = b + \frac{b}{8} + \frac{b}{4} + \frac{b}{2}$, or $\frac{a}{8}$ = 1 + : 125 + : 25 + : 5, or to 1 : 875.
8 × 8	9	$a = 8b + b + \frac{8b}{4} + \frac{8b}{2}$, or a = 15 : 000. Wherefore $a = 15$.

So that the Import of the Question is only to find a Number, from which taking half and half an Unit; and from the Remainder its half and half an Unit: The last Remainder shall be equal to 1. 'Tis plain also, That if a fourth time he had left half the Remainder and half an Egg, he would have had nothing left.

Question V.

Acer in Æmonia fugientem valle Lycisca
Insequitur Leporem picta per arva vagum:
Hic decies quinis præcedit passibus, ille
Instat, & exultans per Juga lata ruit:
Dumq; quater saliendo Lepus consurgit in altum,
Hic toties ternis Saltibus evehitur.
At tantum geminis percurrit Saltibus Agri
Interea, quantum conficit ille tribus.
Dic mihi jam Quoties, saltus iterante Lycisca,
Contigit infesto præda petita cani?

Clark's Ought. explicat.

A Hare being 50 Paces before a Greyhound, makes four Leaps to the Dog's three; but two Leaps of the Dog's are as much as three of the Hares. How many Leaps must the Greyhound take to catch the Hare?

Let $50 = b$.

$4 : 3 :: r : s$

$2 : 3 :: m : n$,

For the Number of the Dog's Leaps sought, put a.

$$\left| \; 1 \; \right| \; s : r :: a : \frac{ra}{s}, \text{ \&c.}$$

Say, As the Number of the Dog's Leaps, to those of the Hare in any time: So will all the Dog's Way be to all the Hare's, after he began to course her.

ra

	2	$\frac{ra}{s} + b$, or $\frac{ra+bs}{s} =$ the whole Number of the Paces the Hare went.
	3	$m : n :: a : \frac{sb+ra}{s}$.

This Proportion ſhews you, that as 2 is to 3, ſo is the whole Number of the Dog's Leaps to that of the Hare's.

	4	$na = \frac{sbm + mra}{s}$ becauſe in four Proportionals, the Rectangle of the Extream is $=$ to that of the mean Terms.
$4 \times s$	5	$sna = sbm + mra$.
Tranſp.	6	$sna - mra = sbm$.
$6 \div sn - mr$	7	$a = \frac{sbm}{sn - mr} = 300$.

Queſtion VI. *In three Bags there is a certain Qanntity of Pounds Sterling. The Sum of the Pounds in the firſt and ſecond Bag, is* 20 l. *The Sum of the Pounds in the ſecond and third Bag, is* 48 l. *And the Sum of the Pounds in the firſt and third Bags, is* 44 l. *What Number of Pounds was in each?*

Let a e and y be put for the Quantities ſought: And $20 = b$: $58 = c$ and 44 d.

Then

Then by the Queſtion.

$$\begin{cases} a + e = b \\ e + y = c \\ y + a = d \end{cases}$$

		$a + e = b$	1
		$e + y = c$	2
		$y + a = d$	3

	4	$e = b - a$,	by Tranſpoſition of a.
	5	$y + b - a = c$,	because $b - a = e$, and $+ y = c$.
	6	$y - a - b + c$,	by Tranſpoſition of $b - a$.
3, 6,	7	$a - b + c + a = d$,	That is, $2a - b, + c = d$.
			Therefore,
Tranſp.	8	$2a = d + b - c$.	
	9	$a = \dfrac{d + b - c}{2} = 8$	Which is a Canon to find a.
4,	10	Then ſince $b - a = e$: $e = 12$.	
6,	11	And $y = 36$.	

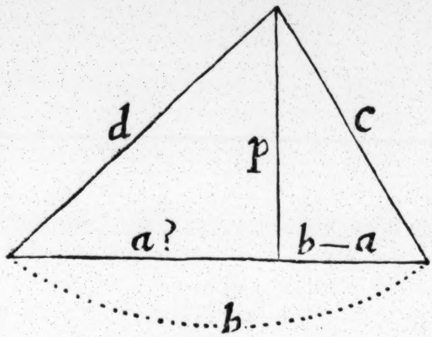
This is called a Queſtion by various Poſition, of which you have many in *Kerſey*, and other Writers,

PROBLEM I.

To determine the Point where a Line perpendicularly let fall from the Vertex, or Top of an Acute-angled Triangle, d b c *ſhall cut the Baſe* b.

Suppoſe it done, and the Figure drawn, call one Segment of the Baſe a; then will the other be $b - a$, and let the Perpendicular be called p.

47, c. I.



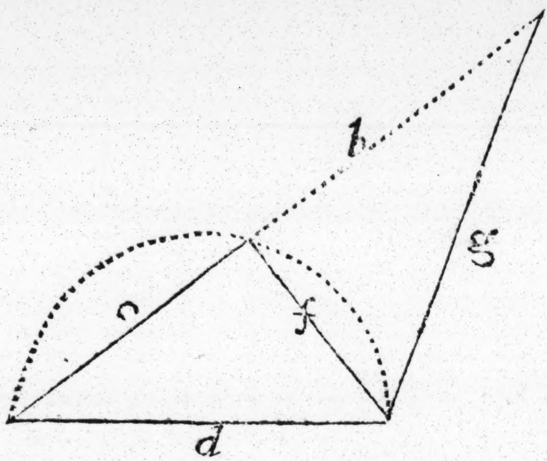
47. c. 1.	1	$pp = dd - aa$, and $pp = cc - bb + 2ba - aa$.
Tranſ. aa	2	Wherefore $dd = cc - bb + 2ba$.
Tranſp.	3	And conſq. $dd + bb - cc = 2ba$.
$3 \div 2b$	4	And therefore $\frac{dd + bb - cc}{2b} = a$.

Wherefore the Length of *a* is found Arithmetically, or by Calculation; for if you add the Square of the Baſe *b*, and of the Side *d* (meaſured on any Scale) together, from the Sum ſubtract the Square of the Side *e*: And then divide the Remainder by the double of the Baſe *b*, the Quotient will give you the Length of *a*, and conſequently determine the Point where *p* will cut the Baſe *b*.

And you may conſtruct the Equation in the fourth Step Geometrically. Thus,

On *d*, the longeſt Leg of the given Triangle, make a Semicircle; and in it apply *c*, the other Leg, drawing alſo the Line *f*.

Then



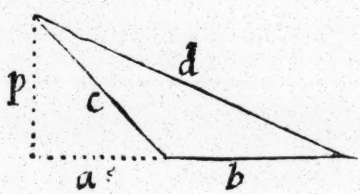
Then will $ff = dd - cc$ produce c, till the Part without the Semicircle be equal to b, the Base of the given Triangle: And draw the Line g.

Then will $gg = ff + bb = dd - cc + bb$, and since in the fourth Step, this last Quantity is to be divided by $3b$, make, as $2b : g$, which is $=$ to $\sqrt{:dd - cc + bb)} :: g$ to a fourth Proportional, which will be a. And consequently a will be found Geometrically.

PRO-

PROBLEM II.

To find the Point without an Obtuse angled Triangle d c b, *where the Base* b *being produced, shall meet with a true Perpendicular* p *let fall from the Vertex of the Triangle.*



Suppose it done, and all things noted as you see in the Figure. Therefore,

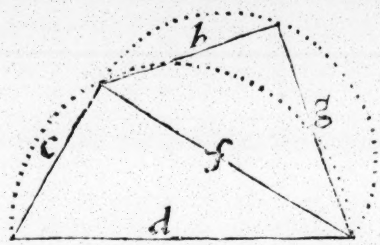
47. *e.* 1.	1	$pp = cc - aa$, and $p = dd - aa - 2ba - bb$.
Transp. aa	2	$cc = dd - 2ba - bb$. And,
By Transp.	3	$cc + 2ba = dd - bb$. And,
	4	$2ba = dd - bb - cc$. Wherefore,
$5 \div 2b$.	5	$a = \dfrac{dd - bb - cc}{2b}$..

And consequently a is found by Calculation.

The Geometrical Construction.

On d the longest Leg of the Triangle given, describe a Semicircle, and in it apply e the other Leg: Drawing also the Line f, so will $ff = dd - cc$: Then on f describe also another Semi-

circle, and therein apply b the Base of the Triangle given: Drawing likewise the Line g. Then



will $gg = ff - bb$. That is, $= dd - cc - bb$: And consequently $g = \sqrt{:dd - cc - bb}$. And since the fifth Step of the Equation is divided by $2b$; make, as $2b$ to g :: so g to a fourth Term; which will be a, the Side sought.

Quadratick Equations.

QUadratick *Equations*, are such as retain on the unknown Side, the Square of the Root or Number sought; and are of two sorts.

I. *Simple Quadraticks*, where the Square of the unknown Root is equal to the absolute Number given, as $aa = 36$, $ee = 146$, $yy = 133225$. And for the Solution of these, there needs only to extract the Square Root out of the known Number, and that is the Value of the Root or Quantity sought: Thus the Value of a in the first Equation is equal to 6, in the second $e = 12$ and a little

little more, it being a Surd Root. And in the third Example $y = 365$.

II. *Adfected Quadraticks*, are ſuch as have between the higheſt Power of the unknown Number, and the abſolute Number given, ſome intermediate Power of the unknown Number, as $aa + 2ba = 100$.

And this Equation is properly called *Adfected*, becauſe the unknown Root a is multiplied into the Co-efficient $2b$.

The Original of *Adfected Equations*, the Incomparable *Harriot* thus derives: Let a be $= + b$, or $a = - c$, then by Tranſpoſition will $a - b = 0$, and $a + c = 0$. And then multiplying one by another, the Product is $aa - ab + ca - bc = 0$.

And this he properly calls an *Original Equation*. From which, or others of the ſame kind, Tranſpoſing bc over to the other ſide with a contrary Sign, he gains ſuch an Equation as this, $aa - ab + ca = bc$, which he calls a *Canonical Equation*.

And from hence, by putting Examples in all Caſes he ſhews, that every poſſible *Quadratick Equation* hath *two Real Roots*, according to the Dimenſions of the higheſt Power; as being made up by the Multiplication of two Simple Equations. And that theſe two Roots may be either both Affirmative, or both Negative, and that ſometimes they are equal to each other, and ſometimes not. And from hence he finds, That the abſolute Number bc, is always the Rectangle of the two Roots b and c (or of the two Values of a:) And that if it have a Poſitive Sign, the two Roots have *like* Signs, but if a Negative one, *unlike*.

And, That the Co-efficient of the middle Term is always the *Aggregate of both the Roots with contrary Signs*; and consequently their *Difference*, when without its Sign. See more in his *Second Section*, and *Wallis*'s *Algebra*, p. 132, &c.

And when in such kind of Quadraticks as these, the Indexes or Exponents of the Dimensions of the unknown Root are in Arithmetical Proportion; that is, as in this Equation, $aa + 2ab = 100$, the Index of aa is 2, the Index of $2ba$ is 1, and the Index of 100 is 0; then may the Root be easily found by the following Method.

All Equations of this Rank will be in one of these three Forms.

$$\left.\begin{array}{l} aa + ad = R \\ aa - ad = R \\ as - aa = R \end{array}\right\}$$

* *Some make four Forms, but at long run it comes to the same thing.*

In all which Forms, R, the absolute Number given, is a Rectangle, or Product made out of the two Quantities or Roots sought, a Greater and a Lesser.

Of which, in the *First Form*, where all is Affirmative, the Co-efficient d is the Difference between those two Quantities or Roots, and a is the Lesser of them; as is plain, if you suppose the two Roots (as *Oughtred* doth) to be a the Greater, and e the Lesser. For then let $d = x$ be the Difference between them; so that $e + x = a$, if then you multiply each Part by e, it will be $ee + ex = ae$; from whence it appears also plainly, that ae is equal to R, the absolute Number given, or equal to the Rectangle of the two unknown Roots

Roots a and e, of which in this Form the Co-efficient x or d is equal to the Difference between them, and e is the Leſſer of them.

In the *Second Form*, The Co-efficient d is the Difference of the two Roots as before, but *a there* repreſents the Greater of them, as is plain, by putting (becauſe the Sign is Negative) $a - x = e$, and multiplying each Part by a, it produces $aa - ax = ae$, the ſecond Form, where x or d the Co-efficient is the Difference of the two unknown Roots; and a repreſents the Greater of them.

In the *Third Form*, where the higheſt Power is Negative, the Co-efficient s is the Sum of the two Quantities or Roots ſought; and a the Affirmative Root ſought may be either the Bigger or the Leſſer of them. For let (becauſe the higheſt Power is Negative) $z - a = e$: Then multiplying both by a, it will be $za - aa = ae = R$; or if $z - e$ had been put equal to a, then it would have been $ze - ee = ae$, by multiplying all by e.

So that this Method ſhews the Original Conſtitution of theſe Forms, and the Nature and Office of each Member of them.

From all which may be found this General Canon for the Solution of *Quadratick Equations*, according to this Method.

Multiplying the abſolute Number by 4, and to the Product add the Square of the Co-efficient, then extract the Square Root of that Sum; which Root ſhall be the Sum of the two Numbers ſought. Then to or from the half of that Root, add

add or ſubtract half the Co-efficient, and the Sum and Remainder are the two Roots requi-red.

For the particular Solution of *Adfected Quadraticks*, there are three Ways.

I. *That of* Oughtred, *who proceeds in this Method.*

In all the three Forms, there is given either the *Rectangle* and *Sum*, or the *Rectangle* and *Difference* of the two unknown Quantities; whence 'tis very eaſie to find either the *Difference* in the former, or the *Sum* in the latter *Caſe*; and then having the *Sum* and *Difference* of any two unknown Quantities, the Quantities themſelves will ſoon be known.

Thus in the firſt Form. Let $aa + da = R$.

Here is given R, the Rectangle of the Roots, d their Difference; and 'tis known that a repreſents the Leſſer of them. Let S ſtand for the Sum to be ſought.

Let $a + e = s$, and $a - e = d$.

Then $aa + 2ae + ee = ss$, and $aa - 2ae + ee = dd$.

Subduct the latter from the former.

$$\begin{array}{r} aa + 2ae + ee \\ aa - 2ae + ee \\ \hline 4ae = 4R \end{array}$$

Wherefore $SS - dd = 4R$. And therefore,

You

You may, by *Simple Algebra*, find that $4R = SS - dd$: and consequently that $4R + dd = SS$, and therefore S is known; and then having S and d, a the lesser Root will be known too, for $\frac{1}{2}S - \frac{1}{2}d = a$.

Again, in the second Form. Let $aa - ad = R$.

Here d and R (as before) the Difference and Rectangle of the two Roots are given; and a the greater of them; wherefore 'tis easie to find S the Sum, and then $\frac{1}{2}S + \frac{1}{2}d = a$.

In the third Form. Where $Sa - aa = R$.

There is given the Co-efficient S = Sum of the unknown Roots, R the Rectangle between them; and a may be either the bigger or the lesser of them: Here therefore to find d the Difference.

Because $SS - dd = 4R$, therefore $SS - 4R = dd$, and consequently d is known; and then $\frac{1}{2}S + \frac{1}{2}d =$ greater, and $\frac{1}{2}S - \frac{1}{2}d =$ lesser.

II. *The Solution of Adfected Quadratick Equations, by the Method of compleating the Square.*

Which is by Mr. *Harriot* thus: Since in every one of the three Forms of *Quadraticks*, one Quarter of the Square of the Co-efficient will make the unknown Side of the Equation, a compleat Square, whose true Root will be $a + \frac{1}{2}d$, (or whatever Letter else be the Co-efficient.) 'Tis plain by this means

means an *Adfected Quadratick Equation* may be reduced to a *Simple* one.

Wherefore,

In the firſt Form, when all the Species are Affirmative,

Let $aa + da = R$.

If $\frac{1}{4}dd$ be added to the unknown Side, it will be a perfect Square $aa + da + \frac{1}{4}dd$, whoſe true Root is $a + \frac{1}{2}d$.

Add then, $\frac{1}{4}dd$ to R, and $R + \frac{1}{4}dd$ will be a perfect Square Number and known; whoſe Square Root extracted in Numbers, will be equal to $a + \frac{1}{2}d$: And conſequently, a will be equal to that Root, when $\frac{1}{2}d$ is taken from it, and ſo a will be known.

The Practical Rule is this.

To the abſolute Number, add $\frac{1}{4}$ of the Square of the Co-efficient, (or the Square of half the Co-efficient) and extract the Root of the Sum: Then from that Root found in Numbers, ſubtract $\frac{1}{2}$ the Co-efficient, and the Remainder is a, the leſſer of the two Roots, or Values of a.

EXAM-

EXAMPLE.

$$aa + da = R$$

Or, $aa + 16a = 36$.

To $36 = R$.

Add $64 = \frac{1}{4}dd$.

$\sqrt[2]{:100} = 10 = a + \frac{1}{2}d$.

But $\frac{1}{2}d = 8$

Therefore $2 = a$.

In the ſecond Form, Let $aa - da = R$.

Proceed in all reſpects as in the firſt Form, only you muſt at laſt add half the Co-efficient to the Root extracted out of the Abſolute Number, inſtead of taking it from it, as before; becauſe here a repreſents the greater Root: And thus, If $aa - 16a = 36$, a will be found $=$ to 18.

In the third Form. Let $Sa - aa = R$.

Here, becauſe the higheſt Power is Negative, 'tis impoſſible any ſuch Root can be found that will produce $-aa$; wherefore you muſt imagine all the Signs changed, and it will ſtand thus, $-Sa + aa = -R$; or putting the higheſt Power firſt, $aa - Sa = -R$.

In this Form, the Co-efficient is the Sum of the two Roots, and a may be either of them.

 And

And here the abſolute Number is ſo determined, as that it cannot be greater that the Square of half the Co-efficients: Wherefore,

The Practical Rule is this:

From the Square of half the Co-efficient, take the abſolute Number given, and extract the Square Root of the Remainder; which Root either added to, or ſubtracted from half the Co-efficient, will give accordingly the greater or leſſer Value of *a*.

Thus, If $20a - aa = -36$
Or, $Sa - aa = -R$

From $100 = \frac{1}{4}SS$
Take $36 = R$
$\sqrt[2]{:64} = 8$

Now $10 + 8 = 18$ the greater Root.
And $10 - 8 = 2$ the leſſer Root.

III. *To Solve* Quadratick Adfected Equations, *by taking away the* Second Term.

In any of the three Forms, if the Co-efficient have a Negative Sign, put $e + \frac{1}{2}d$, but if it have an Affirmative Sign, put $e - \frac{1}{2}d$, instead of *a*, the Root of the higheſt unknown Power.

Then will $ee \mp ed + \frac{1}{4}dd = aa.$

Alſo $\mp ed \mp \frac{1}{2}dd = \mp da.$

And theſe two Quantities added together, muſt be equal to the abſolute Number given; and the Equation will become a Simple one.

In the firſt Form, $aa + da = R$, or $aa + 16a = 36.$

Let $e - \frac{1}{2}d = a.$

Then will $ee - ed + \frac{1}{4}dd = aa.$

And $ed - \frac{1}{2}dd = ad.$

Which added together, make $ee - \frac{1}{4}dd = R.$

Therefore $ee = R + \frac{1}{4}dd,$

And conſequently, $e = \sqrt{R + \frac{1}{4}dd}.$

But $e - \frac{1}{2}d = a.$

Therefore $e = a + \frac{1}{2}d.$

Conſequently $a + \frac{1}{2}d = \sqrt{R + \frac{1}{4}dd}.$

Wherefore $a = \sqrt{R + \frac{1}{4}dd} - \frac{1}{2}d.$ *Q. E. D.*

And ſince $a + \frac{1}{2}d = \sqrt{R + \frac{1}{4}dd}$; If each part of the Equation be Squared, there will ariſe,

$$aa + ad + \frac{1}{4}dd = R + \frac{1}{4}dd.$$

Which is the other common Canon for ſolving *Quadraticks*, by adding to each Part the Square

of half the Co-efficient, in order to compleat the Square.

In the ſecond Form, $aa - ad = R$.

Let $e + \frac{1}{2}d = a$.

Then is $ee + ed + \frac{1}{4}dd = aa$.

And $-ed - \frac{1}{2}dd = -ad$.

These added make $ee - \frac{1}{4}dd = R$.

Therefore $ee = R + \frac{1}{4}dd$.

And $e = \sqrt{:R + \frac{1}{4}dd}$.

But $e + \frac{1}{2}d = a$.

Therefore $e = a - \frac{1}{2}d$.

And conſequently $a - \frac{1}{2}d = \sqrt{:R + \frac{1}{4}dd}$.

Wherefore $a = \sqrt{:R + \frac{1}{4}dd} + \frac{1}{2}d$.

Q. E. D.

And ſince $a - \frac{1}{2}d = \sqrt{:R + \frac{1}{4}dd}$; if each Side of the Equation be Squared, you will have,

$$aa - ad + \frac{1}{4}dd = R + \frac{1}{4}dd.$$

Which is the common Canon for ſolving Equations, by compleating the Square.

In

In the third Form, $da - aa = R$.
Which Form must be thus changed,

$$aa - da = -R.$$

Then make as before, $e + \frac{1}{2}d = a$.

And then $ee + ed + \frac{1}{4}dd = aa$.
And $\quad -ed - \frac{1}{2}dd = -ad$.

Whose Sum is $ee - \frac{1}{4}dd = -R$.
Then is $ee = \frac{1}{4}dd - R$.
And $e = \sqrt{: \frac{1}{4}dd - R}$.
And since, $e + \frac{1}{2}d = a$.
$e = a - \frac{1}{2}d = \sqrt{\frac{1}{2}dd - R}$.

Wherefore (because there are two Positive Roots in this Form)

$$a = \sqrt{: \frac{1}{2}dd - R} \pm \frac{1}{2}d.$$

But the Value of a is Ambiguous, and you must generally try both Roots, before you can find which will solve the Question: Whereas in the other two Forms the first a found, will be that required.

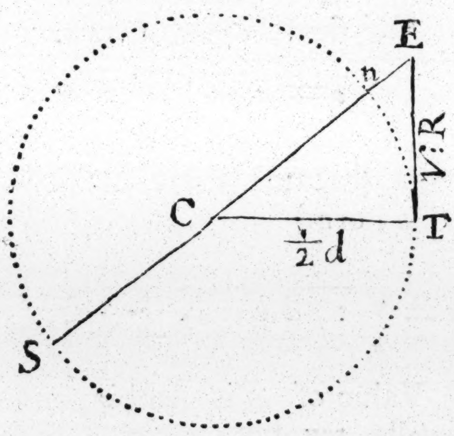
N. B. *In this way of Solving* Quadraticks, *the known Quantity added to, or subtracted from* e, *must be always half the Co-efficient.*

Construction

Conſtruction of Adfected Quadraticks.

THE Conſtruction of *Simple Quadraticks*, you have before under *Simple Equations* : That of *Adfected* ones is eaſily done many ways.

I. In the firſt Form of *Quadraticks*, let $aa + da = R$: Then by the common Method of Solution, $a = \sqrt{:R + \frac{dd}{4}} - \frac{d}{2}$.



Wherefore deſcribe a Circle, whoſe Radius ſhall be CT ½ d, and make the Tangent TE = √ : R, drawing alſo the Secant SCE ; then will

$CE = \sqrt{:R + \frac{dd}{4}}$ (by 47. *e.* 1. *Euc.*) and conſequently $nE = \sqrt{R + \frac{dd}{4}} - \frac{1}{2}d = a$.

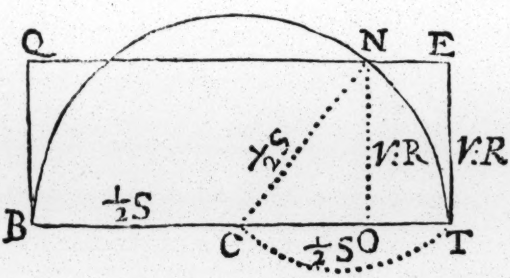
II. In

II. In the second Form, where $aa - da = R$;

a will be equal to $\sqrt{:R + \frac{dd}{4}} + \frac{d}{2}$. And consequently the same Construction and Diagram will serve here, which was used in the first Form: And the Root will be represented by $SE = \sqrt{:\frac{1}{4}dd + R} + \frac{1}{2}d$.

III. In the third Form, where $Sa - aa = R$;

a will be equal to $\frac{S}{2} \pm \sqrt{:\frac{SS}{4} - R}$, and here the Root a hath two real Values; make CT $(= \frac{1}{2}S)$ the Radius of a Circle, and erect the



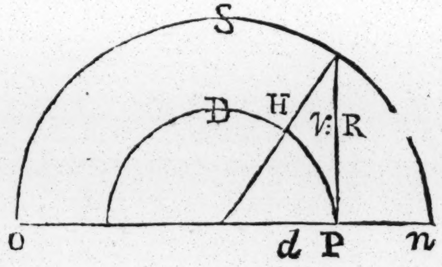
Perpendicular $ET = \sqrt{:R}$, then draw EQ Parallel to CT, and NO Parallel to ET; draw also the Radius CN. Then will (by 47. *e.* 1. *Euclid.*) $CO = \sqrt{:\frac{SS}{4} - R}$, and consequently,

$BO = \frac{S}{2} + \sqrt{:\frac{SS}{4} - R} =$ the greater Root

4, and $OT = \frac{S}{2} - \sqrt{:\frac{SS}{4} - R}$. Or the two Roots will be QN and NE, equal to the former.

Another way of Construction of Adfected Quadraticks.

DR. *Wallis*'s way of Constructing the three Forms of all *Quadratic Equations*, according to M. *Oughtred*'s Method of Solution is this: Draw two Concentrick Circles, and let the Diameter of the greater be called S, and the Diamerer of the



lesser D, the Sum and Difference of the Roots found. Wherefore H and *d* will represent the half Sum and half Difference of the Roots.

Since therefore *Oughtred*'s *Theorem*, as is shewed above, is that $SS - DD = 4R$, Let $\sqrt{R}$ be made a Tangent to the lesser, or a Right Sine to the greater Circle, as you see in the Figure, according as D, or S, is given: And draw also the Hypo-

Hypothenuse H. Then will the Base of the Triangle be d. And $HH - dd = R$ (by 47. *e.* 1.) That is, $\frac{SS}{4} - \frac{DD}{4} = R$. Wherefore by Transposition, $HH = R + dd$, and therefore $H = \sqrt{:R + dd}$: And consequently, if it had been in the first and second Forms, where d and R were given, H will also be found. Or if H had been given, and d required as in the third Form. Since $HH = R + dd$; Therefore $HH - R = dd$: and $\sqrt{:HH - R} = d$: And having thus found H and d, the half Sum and half Difference of the two Roots. Then $H + d$ ($= op$) will be the greater Root a, and $H - d$ ($= pn$) will be the lesser, which will be Affirmative or Negative, according to the Form and Circumstances of the Equation.

A Question and Problems in Adfected Quadratick Equations.

QUESTION.

Two Men have each a certain Number of Crowns, whose Sum subtracted from the Sum of their Squares, leaves $R = 78$: *But their Sum added to the Product of the two Numbers, makes* $39 = S$. *How many Crowns had each?*

For the unknown Sum of the Numbers put $2a$. And for their Difference $2e$: For then the Numbers may be thus noted, $a + e = <$ and $a - e = >$.

 Where

Where > and < signifie the Greater and Lesser.

Then,

	1	$2aa + 2ee =$ Sum of their Squares.
$1 - \overline{2a}$	2	$2aa + 2ee - 2a = R$, by the State of the Question.
$2 \div \overline{2}$	3	$aa + ee - a = \frac{R}{2} : 39 : = S.$
By Transf.	4	$39 - aa + a = ee$, which Step will at last find e.
$\square + 2a$	5	$aa - e + 2a = S$. Their Product added to their Sum.
By Transf.	6	$aa + 2a - S = ee.$
4, 6,	7	$30 - aa + a = aa + 2a - 39$ (S) $= ee.$
By Transf.	8	$78 = 2aa + a.$
	9	$aa + \frac{1}{2}a = 39$ S; which is a Quadratick of the first Form.
Compl. $\square$	10	$aa + \frac{1}{2}a + \frac{1}{16} = 39 + \frac{1}{16}.$
ω	11	$a + \frac{1}{4} = \sqrt{:39 + \frac{1}{16}}.$
	12	$a = \sqrt{:39 + \frac{1}{16}} - \frac{1}{4} = 6.$
	13	Therefore $2a = 12.$

And (a) being known, the Value of (e) will be found from the fourth Step. Where $e = 3$,

Now, by our Supposition at first, the greater Number was $a + e$, that is 9; and the lesser was $a - e$; that is 3: Which Numbers 3 and 9, will answer the Question.

For

For 12, their Sum, taken from 90, the Sum of their Squares, leaves 78; and added to 27, their Rectangle, makes 39.

N. B. *By this Method of putting* a + e *and* a — e *for the two Numbers sought, instead of* a *and* e, *as in the common way; many Questions producing Adfected Quadratick Equations, when that way manag'd, may be solved as easily, and in the manner of Simple Equations. Especially when the Sum and Difference, or Sum or Difference of the Squares of the Quantities sought, are among the* Data.

PROBLEM I.

The Difference of both the Legs of a Right-angled Triangle being given from the Hypothenuse; to find the Sides and the Triangle severally, and to Form it.

b

d

Let the Difference of the lesser Side from the Hypothenuse be (*b*) and that of the greater (*d*.) For the greater Side sought put (*a*).

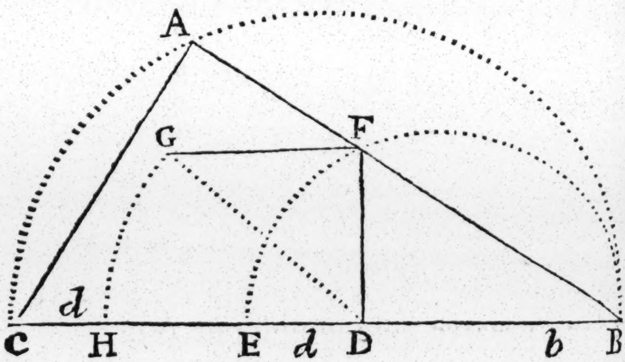
Then will,

	1	$a+d=$ Hypothenuse. And,
$1-b$	2	$a+d-b=$ to the lesser Side.
47 e. 1	3	$aa+2ad+dd:=2aa+2ad-2ab-2bd+dd+bb.$
	4	$aa-2ab-2bd+bb=0$, by Comparison and Transposition of the last Step.

Transp.	5	$aa - 2ab = 2bd - bb$, which is a Quadratick Equation of the second Form.
Compl. □	6	$aa - 2ab + bb = 2bd$.
ω	7	$a - b = \sqrt{:2bd}$
		$a = \sqrt{:2bd} + b$.

The General Construction.

Find a mean Proportional between d and b, which let be DF; to which, place at Right-angles FG = to DF, draw DG, and cut off HD = GD. Then will BH be the greater Side

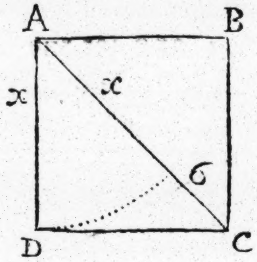


sought. And this being produced to C (so that CH = ED) will give CD (= AC) the lesser Side of the Triangle required. Draw a Semicircle on CB, and apply AB = HB. Then draw AC, and the Triangle is found, which is ABC.

PRO

PROBLEM II.

Having in the Square ABCD, *the Difference between the Sides and Diagonal* = 6, *or* a, *to find the Side of the Square.*



Let the Side ſought be called x, and $6 = a$.

Then $x + a =$ A C the Diagonal.

But (by 47. *e.* 1. *Eucl.*) ACq, $= 2$ADq, or to $2\,x\,x$;

That is $x\,x + 2\,x\,a + a\,a = 2\,x\,x$.

Expunge then $x\,x$ on both Sides, and it will be

$$2\,x\,a + a\,a = x\,x.$$

And then by Tranſpoſition, $x\,x - 2\,a\,x = a\,a$.

Compleat the Square, and it will be

$$x\,x - 2\,a\,x + a\,a = 2\,a\,a.$$

Wherefore $x - a = \sqrt{:2\,a\,a}$,

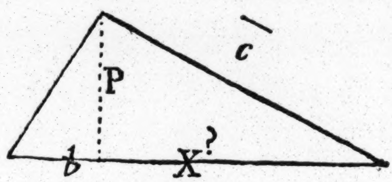
And conſequently $x = \sqrt{:2\,a\,a} + a = 14.48$.

PRO-

PROBLEM III.

Given one Segment of the Baſe of a Right-angled Triangle, as alſo the Side of the Triangle adjacent to the other Segment of the Baſe; 'tis required to find the reſt, and to form the Triangle.

Suppoſe it done; and let the Segment b, and the Side c, be both known or given. Let, x the



other Segment of the Baſe, be ſought; which is all that is neceſſary to ſolve the *Problem.*

Here therefore, ſince P is ſuppoſed to be a true Perpendicular.

1 $cc - xx = pp$, 47 *e.* 1. *Euclid.*

2 And becauſe the Angle at the Top is a Right one, therefore $pp = bx$, which gives another way of expreſſing pp. So that,

3 $cc - xx = bx$, and conſequently by Tranſpoſition.

4 $cc = xx + bx$, which is an Adfected Quadratick of the Firſt Form. Wherefore,

5 $cc + \frac{bb}{4} = xx + bx + \frac{bb}{4}$ by compleating the Square. And,

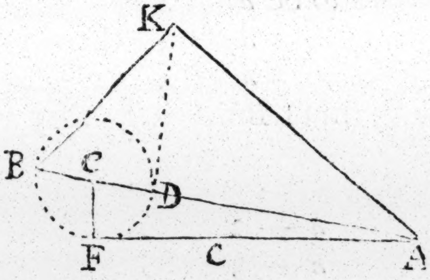
6 | $\sqrt{cc + \frac{bb}{4}} = x + \frac{b}{2}$, by Evolution.

Laftly,

7 | $\sqrt{cc + \frac{bb}{4}} - \frac{b}{2} = x.$

Geometrical Conftruction.

Join together at Right Angles e F $= \frac{1}{2}b$ and FA $=$ C. Then with the Radius e F defcribe the Circle B F D, and thro' the Centre e draw the Line ADB. Erect then at D the Perpendicular



DK, which limit, by defcribing a Semicircle on B A: That Semicircle fhall cut the Perpendicular in the Point K, the Vertex of the Triangle required, whence draw the two Legs B K and K A. So is B K A the Triangle fought.

Of

Of Cubick and Biquadratick Equations.

MR. *Harriot* ſhews the Original of a *Cubick Equation* to be derived either from three Lateral or Simple Equations, reduced firſt to the Form of *Binomials*, and then multiplied continually into each other; or elſe from one Quadratick multiplied by a Lateral.

Whence he deduces that all *Cubick Equations* have Real or Imaginary, 3 Roots; or as many as are the Dimenſions of its higheſt Power.

Thus to form a *Cubick Equation*, let its three Roots or Values to be

$$\left.\begin{array}{l} a = 2 \\ a = 3 \\ a = 4 \end{array}\right\} \text{then by reducing}$$

them to this Form of *Binomials*, they will ſtand thus,

$$a - 2 = 0$$
$$a - 3 = 0$$
$$a - 4 = 0$$

And theſe 3 *Binomials* multiplied continually into one another, do produce this Equation $aaa - 9aa + 26a - 24 = 0$, or $a^3 - 9aa + 26a = 24$. Which *Cubick Equation* would have been produced alſo by multiplying the Quadratick $aa - 5a + 6 = 0$, by $a - 4 = 0$.

In like manner he ſhews the Derivation of a *Biquadratick Equation*, to be either from four Simple

Simple or Lateral Equations, reduced as above, to the Form of *Binomials*, and continually multiplied into one another: Or elſe from a *Cubick* into a *Lateral*, one Quadratick into another, or a Quadratick multiplied by 2 Laterals. Wherefore he ſaith, every Biquadratick will have Real or Imaginary, four Roots, agreeable to the Dimenſions of its higheſt Power.

Thus if the former Cubick $a^3 - 9aa + 26a - 24 = 0$, be multiplied by $a + 5 = 0$, there will ariſe this *Biquadratick Equation*, $a^4 - 4a^3 - 19aa + 106a - 120 = 0$; that is, $a^4 - 4a^3 - 19aa + 106a = 120$.

From which Original of theſe *Cubick* and *Biquadratick Equations*, 'tis plain, That as ſoon as you can diſcover the Value of any one Root, you may depreſs the Equation a Dimenſion lower, by dividing it by ſuch Root reduced to the Form of a Binomial as above. Thus, If you find that one Root, or one a is $= 2$, then divide the laſt Equation by $a - 2$, and it will bring it down to a *Cubick*; and that *Cubick*, being again divided by $a - 3$, $a - 4$, or $a + 5$, will be depreſſed into a *Quadratick*, &c. And this is ſometimes of good Uſe to diſſolve *Compound Equations* into their Components, as hath been ſhewn by *Des Cartes*, *Hudd*, and others.

From this Method of Compoſition of theſe Equations, 'tis alſo apparent, of what Members each of the Co-efficients are made up. For,

I. *The Co-efficient of the ſecond Term, is always the Aggregate of all the Roots under contrary Signs.* Thus, In the *Cubick Equation* above mentioned, the Co-efficient 9, is the Sum of 2, 3, and 4, with the

the Negative Sign. And 4 the Co-efficient of the second Term in the *Biquadratick Equation* above mentioned, is the Aggregate of 2, 3, 4 and —5; that is, 4 with a Negative Sign. Wherefore it follows, That if all the Negative Roots, secluding their Signs, be equal to all the Affirmatives; (tho' not each to each respectively) then will the Second Term quite vanish out of the Equation, and be *wanting*, as 'tis call'd; because the Negatives and Affirmatives do mutually destroy each other: And *vice versâ*, whenever the second Term is *wanting* in one of these Equations, the Roots are thus equal, and have contrary Signs.

II. The Co-efficient of the Third Term is the Aggregate of all the Rectangles made by the Multiplication of every pair of the Roots, as often as they can be taken; which in a *Cubick* is 3, in a *Biquadratick* 6, in an Equation of the fifth Power 10, &c. according to the Order of Triangular Numbers.

Thus in the Third Term 26 *a* of the *Cubick Equation* before mentioned; 26 the Co-efficient is the Aggregate of 6, 8 and 12, the 3 Rectangles of the Roots 2, 3 and 4.

And here if all the negative Rectangles (secluding their Signs) are equal to all the Affirmative ones, they will destroy one another, and so the Third Term will vanish, or be wanting.

III. The Co-efficient of the fourth Term is the Aggregate of all the Solids made by the continual Multiplication of all the Ternary's, or every three of such Roots so Signed, &c. and so on *ad Infinitum*.

IV. As

IV. As in *Quadraticks* the absolute Number given is always the Rectangle of the 2 Roots, or Values of *a*: So in *Cubicks* 'tis always the *Solid* made by continual Multiplication of all the three Roots one into another; and in *Biquadraticks*, of all the four Roots, *&c.*

The Resolution of Cubick and Biquadratick Equations.

As to the Resolution of these kinds of Equations, I shall in this short Treatise make no mention of *Kersey*'s way of finding all the just Divisors to the last Term, or absolute Number given; (*See* Kersey'*s Algebra, Vol.* 1. *Book* 2. *Chap.* 10. *Sect.* 9.) Nor of the Common Rules of *Cardan* for *Cubicks*; where the second Term must first be taken away; because at the best they are not Perfect; and also because they are very tedious and troublesome.

I would rather advise the Learner to make Use in ordinary Cases of the general Method of *Stevinus*, mention'd by *Kersey* in the same Chapter: For tho' that be but a tentative Way, and comes to the Truth only by frequent Trials; yet when 'tis made familiar by Practice, and Experience hath taught him how to judge of the Limits of Equations, it will expeditiously enough discover one true Root (if such there be) in almost any kind of Equation; whether having all its *Parodick*

Degrees, or Terms, or not. And when one true Root is found, you may (as is ſhewn above) depreſs the given Equation by it (if it be a Regular one) one Degree lower, and by that means eaſily diſcover the other Roots.

Of this Method take a few Examples.

EXAMPLE I.

Suppoſe the former Cubick Equation *given*, $a^3 - 9aa + 26a = 24$.

Firſt, I will imagine $a = 1$, and working according to the Equation, I find that $27 - 9$ $(= 18)$ $= 24$. Wherefore I conclude a is greater than 1. I try again, and ſuppoſe $a = 2$. Then will $8 + 52 = 36$ (*i.e.* $60 - 36$) $= 24$. Which Anſwers my Deſire, and gives me one real Value of a the Root in this *Cubick Equation*: After which I may either divide the given Equation by $a - 2$, which will bring it down to a *Quadratick*; or I might have proceeded further in the ſame Method, and have found alſo, that 3 and 4 would have been the other 2 real Roots.

EXAMPLE II.

Suppoſe $aaa - 22aa + 157a = 360$.

I make trials with 1, 2, 3 and 4, and find them all too little; wherefore I imagine $a = 5$. Therefore according to the Equation, aaa $(= 125)$ $+ 157a$ $(= 785)$ $- 22aa$ $(= 550)$ $= 360$; which I find is exactly: And thence I conclude, that 5

is

is one true Root of that Equation. Dividing then the Equation $aaa - 22aa + 157a - 360 = 0$ by $a - 5$. I reduce it to this Quadratick $aa - 17a + 72 = 0$. That is, $aa - 17a = 72$, whose two Roots are 8 and 9: Wherefore I conclude 5, 8 and 9, to be the three Roots of the Equation given.

If this irregular Equation were proposed, (where also the absolute Number is a Fraction) $xxxx + 50x = 184638.6801$. I can discover at first Sight almost, that x must be at least equal to 10, and trying with 10, I find it too little; but trying with 100, I find that much too great: Proceeding then again, I find 30 too much; I try 20, and I find that something too small; but 21 I find too big: wherefore I know x must be = to 20, with some Decimal Fraction annexed. And at last I discover 20.7 to be the very Root sought: For Multiplying *that* according to the Equation, it will produce $xxxx + 50x = 184638.6801$.

But for an Universal Method of Extracting the Roots out of all manner of Equations, whether *Pure* or *Adfected*, in Numbers, there must be recourse had to that of an *Infinite* or *Converging Series*. Which, I believe was first hinted by the Incomparable Sir *Isaac Newton*, and afterwards pursued very fully by Mr. *Ralphson*; and from him by Dr. *Lagney*; then the Sagacious Dr. *Halley* took the Matter into Consideration (in *Philosoph. Transact.* N. 210. *A.D.* 1694) Demonstrating the Reason of Dr. *Lagney*'s Rules; and carrying the Thing much farther, by an Universal Method of his own. Which because 'tis there largely delivered, Illustrated with Examples, and Explained by proper Notes and Observations: And because

also

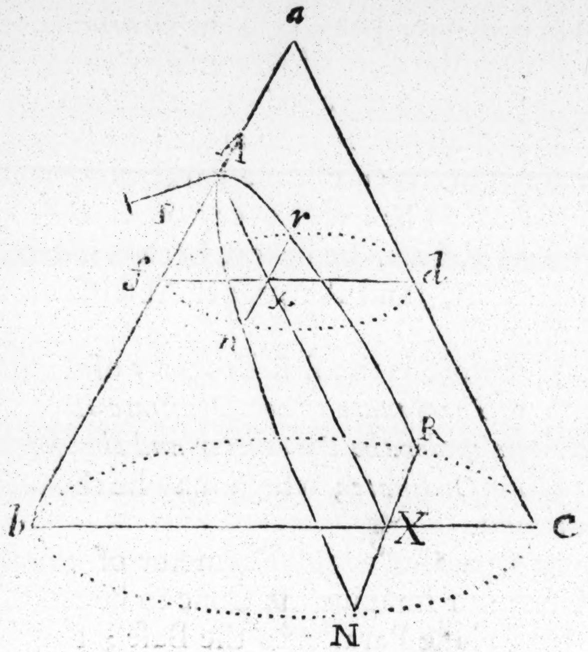
also the Ingenious Mr. *Wells* of *Oxford*, hath given a very good Account of it in his *Element. Arith. Numer. & Specios.* I shall not here stay to explain, but must refer the *Learned Reader* thither. And this I the rather chuse to do, because Mr. *Ward* also in his *Compendium of Algebra*, hath largely insisted on this Subject, in our own Language; hath very much improved on Mr. *Ralphson*'s and Dr. *Halley*'s Foundation, and given a Variety of Theorems and Examples; with very useful *Contractions* in the several Methods of Operation.

I shall therefore next proceed to give you the Geometrical Construction of these kinds of Equations, by which *all their real Roots* will be most easily and readily found.

Construction of Cubick and Biquadratick Equations.

SInce the Construction of these kinds of Equations is done by the Help of the *Parabola*: It will be necessary first to explain what is meant by that Word, and to shew you those Properties of it, which are made use of in these Constructions.

If then a Cone, as *a b c* be cut thro' its Axis the Section will be a Triangle, as *a b c*, and if in the Plane of that Triangle you draw a Line, or A *x* X parallel to either side of the Cone; as suppose here to *a c*: And then in the Plane of the Circular Base of the Cone, erect X N Perpendicular



cular to the Diameter. I say, If you imagine the Cone to be now cut by another Place, according to the two Right-Lines AX, XN, the Section arising from thence, N *n* A *r* RN is called a *Parabola*.

The Point A is called the *Vertex* of the Section, or the *Vertex* of the *Parabola*.

Any Right-Line, as *r x*, or RX; *n x*, or NX; applied at Right Angles to AX (which is called the Axis of the *Parabola*) is called an *Ordinate*: And the Part of the Axis intercepted between the Vertex A and the Ordinate, is called the *Abscissa*: By some, the *Intercepted Axe*.

And if you imagine a Right-line found, as PA, which shall be a third Proportional to the *Abscissa* A *x*, and *Ordinate* *r x*; and which shall be

placed

placed at the Vertex A, Perpendicular to the Axis, or which is all one, parallel to the *Ordinates*: That Line is called the *Latus Rectum*, or *Parameter*.

And the Nature of this *Curve*, the *Parabola*, is such; That if any one Ordinate, as $x\,r$ be a mean Proportional between P A the Parameter, and A x the Abscissa: Then will every other Ordinate, wheresoever drawn, be so too, between the same Parameter P A, and the proper Abscissa to that Ordinate.

This therefore is the first Property of the Parabola; that, *The Square of any Ordinate is equal to the Rectangle under the Parameter and the Abscissa, proper to that Ordinate*; which may be thus easily demonstrated.

Draw $f\,d$ parallel to the Diameter of the Base of the Cone; or imagine the Cone to be cut there again, by a plane Parallel to the Base; then will that Section be a Circle.

1 'Tis plain $x\,d$ will be $=$ X c, as being opposite Sides of a Parallelogram.

2 The Square of X R $=$ Rectangle b X c, and the Square of $x\,r$ $=$ Rectangle $f\,x\,d$, from the Nature of the Circle.

3 A X : X b : : A x : $x f$, by Similar Triangles.

4 Wherefore if X b and $x f$ be multiplied by one and the same Length, the Proportion will continue, and it will stand thus, A X : X b × X c : : A x : $x f$ × $x\,d$.

5 That is in other Words, A X : X R q; : : A x : $x\,r\,q$; by Step 2; and therefore alternately, A X : A x : : X R q; : $x\,r\,q$;. Which by the by, shews you also another general

general Property of the *Parabola* ; that, *The Squares of the Ordinates are to one another as the Abſciſſa.* Vid. Mydorg. Conicorum Lib. 1. Theor. 7.

6 If four Quantities be proportional, they may be expreſſed Fraction-wiſe thus, $\frac{XRq;}{AX} = \frac{xrq;}{Ax}$. And then either of thoſe Quantities will expreſs the *Parameter* or *Latus Rectum* : Let each therefore be called P. Wherefore ſince $\frac{XRq;}{AX} = P : P \times AX$ will be equal to $XRq;$: and $P \times$ by $Ax = xrq;$

7 Wherefore $P : XR :: XR : AX$, and $P : xr :: xr : Ax$.

That is in Words, *The Ordinate is a mean Proportional between the Parameter and the Abſciſſa.*

Or, *The Square of the Ordinate is equal to the Rectangle under the Parameter and Abſciſſa.* Which is the firſt Property of the *Parabola* below to be made uſe of.

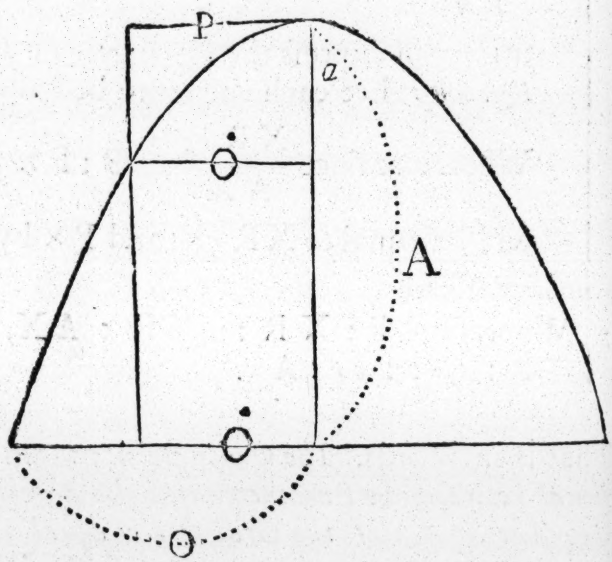
The ſecond Property of the *Parabola*, here to be obſerved, is this ; that *The Parameter is to the Sum of any two Ordinates, as their Difference is to the Difference of the Abſciſſa.*

Which Property, now commonly called by the Name of *Baker*'s Property, was unknown to the Ancients, and was diſcovered by Mr. *Strode* of *Maperton*, who communicated it to Mr. *Baker*, as

he Ingenuouſly owns. Tho' Mr. *Baker* applied it with great Advantage to the Conſtruction of *Cubick* and *Biquadratick Equations*, having all their Terms, as we ſhall ſhew below.

This Property is thus demonſtrated very briefly.

Let p be the Parameter, O and $\dot{O}$ two *Ordinates* and A and a their proper *Abſciſſa*.



I ſay, That $P : O + \dot{O} :: O - \dot{O} : A - a$.

For,

1 $PA = OO$ and $Pa = \dot{O}\dot{O}$, by the first Property of this *Curve*.

2 Wherefore $PA - Pa = OO - \dot{O}\dot{O}$, and then by only

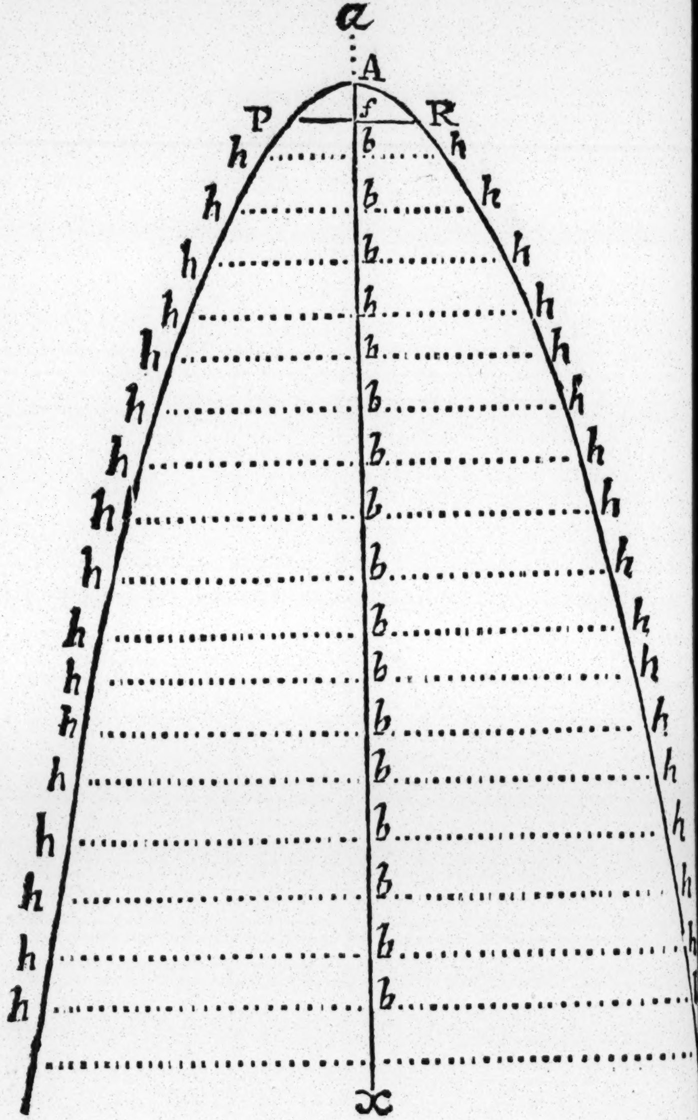
3 Reſolving that Equation into Proportionals it will ſtand thus;

$$P : O + \dot{O} :: O - \dot{O} : A - a. \quad Q.E.D.$$

Becau-

Becauſe in the Conſtructing of Equations this vay, there will be frequent (or rather conſtant) ccaſion to deſcribe Parabola's proper for the E-uation, (for tho' Mr. *Baker* truly ſaith, It may be lone by one and the ſame Parabola, yet whoever vill apply his Theory to Practice, will find great rouble and difficulty in it:) For this Reaſon, I ay, 'tis very convenient to know readily how to lraw a true Parabola on a Plane by a Scale and Compaſs; which is moſt eaſily and expeditiouſly lone thus.

Firſt, Draw the Right-line *a* A *f* X, repreſent-ng the Axis of the deſigned *Parabola*. Then, vhatever the *Latus Rectum*, or *Parameter* of it be, tho' by the by 'twill be beſt, either to make it xactly an Inch, half an Inch, *&c.* from a good DecimalScale, unleſs its Length be determined by one of the *Data* in the Equation, as it often may dvantageouſly be) whatever therefore be the Length of the *Parameter*, ſet one half of it down-wards from *a* to *f*. So ſhall *f* be that Point, which s called the *Focus* of a *Parabola*: Then Biſſect *f a*: And that determins the Point A, which is the *Vertex* of the *Parabola* Next ſet PR (= to the *Parameter*) at Right Angles to the Axis in the Point *f*, which will give P and R, two Points, hro' which the *Curve* of the *Parabola* muſt paſs. Draw then as many Parallels as you pleaſe to PR, lownward from *f* (which is eaſily and ſpeedily lone by a good parallel Ruler) as *b b b*, *b b b*, &c. ſo ſhall the Diſtance *a b*, ſet from *f*, cut and imit every proper correſponding Parallel, and give the Point *b* on each Side the Axis, thro' which the *Curve* of the *Parabola* will paſs.



See the Demonstration of this in Sturm. Mathemat. Enucleat. *and in* Baker's *Geometrical Key*, Pag. 5.

Th

The firſt that attempted the general Conſtruction of theſe Equations, was the Famous *Des Cartes*, who in the third Book of his excellent Geometry, gives a Method by means of a Parabola and a Circle, to conſtruct and find the real Roots of all Equations not exceeding four Dimenſions. Which Method was however not perfect, becauſe it would conſtruct only ſuch Cubick and Biquadratick Equations as had their ſecond Term firſt taken away.

However, becauſe this was it which gave Riſe to *Baker*'s excellent Rule, and to whatever Improvements have been ſince made in it; and becauſe here is laid the firſt Foundation, Reaſon, and Demonſtration of the whole Matter; I ſhall begin with a ſhort Account of it.

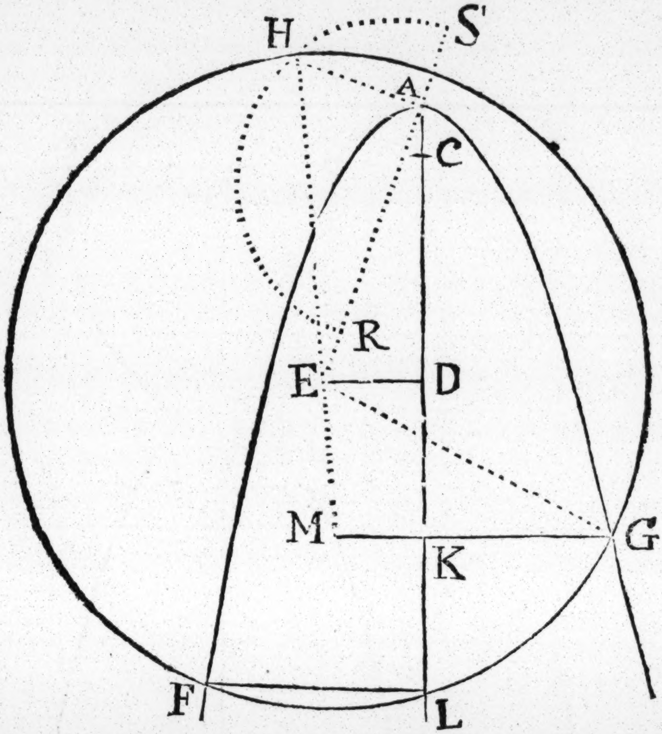
When the ſecond Term is taken away, or wanting, he reduces all *Cubick Equations* to this Form $Z^3 \ast apz : aaq = 0$: And all *Biquadraticks* to this, $Z^4 \ast apzz : aaaz : aaar = 0$, Where a repreſents the *Latus Rectum* or *Parameter* of any given *Parabola*; and is ſuppoſed $= 1$. That ſo its Powers may produce no trouble in the Operation. By which means the former Equations will be in this Form, $Z^3 \ast pz : q = 0$, and $Z^4 \ast pzz : qz : r = 0$. Where z is the unknown Root ſought; p the known Part of the *Third Term*, or a known Number multiplied into the Square of z: q the known Part of the *Fourth Term*, or another known Number multiplied into z, and r (if it be a *Biquadratick*) is the *Fifth Term*, or abſolute Number given. But if the ſecond Term had not been wanting, that would have been p, the third q, the fourth r, &c. and in the *Cubick* q is the abſolute Number.

Let

Let then any Parabola, as F A G, be supposed to be described, whose Axis is A D L, and its Parameter $a = 1$.

First, Take A C equal to half a; so that the Point C will always be within the Parabola. Next in the Axis, (downwards from C, if p have a Negative Sign, but upwards in the Axis produced, when p hath a Positive Sign) take $CD = \frac{1}{2}p$. Then from the Point D thus found, (or from C if the known Part of the third Term q be also wanting in the Equation) erect a Perpendicular to the Axis, as D E, and make it equal to half q: Which Perpendicular D E, must be on the Right-hand of the Axis, if q have a Negative Sign: But towards the Left if it be $+ q$. After which describing a Circle on the Center E, with the Radius E A, it will (if the Equation were only a *Cubick* one) cut the Parabola in as many Points as the Equation hath true Roots: And the Affirmative ones will be Perpendiculars or Ordinates, let fall from the Curve to the Axis on the Right-hand, and the Negative ones such, so let fall to the Axis on the Left-hand.

But if the Equation be a *Biquadratick*, there is something farther necessary to find the Radius of the Circle. For then the fourth Term r being there, and having a Positive Sign, take downwards from A the Vertex of the Parabola on the Line AE, which suppose drawn $AR = r$, and produce RA, till AS become equal to the Parameter, or $= a = 1$. Then make R S the Diameter of a Circle, and at A erect A H Perpendicularly; it shall cut the Semicircle in H: So that H E shall be the Radius

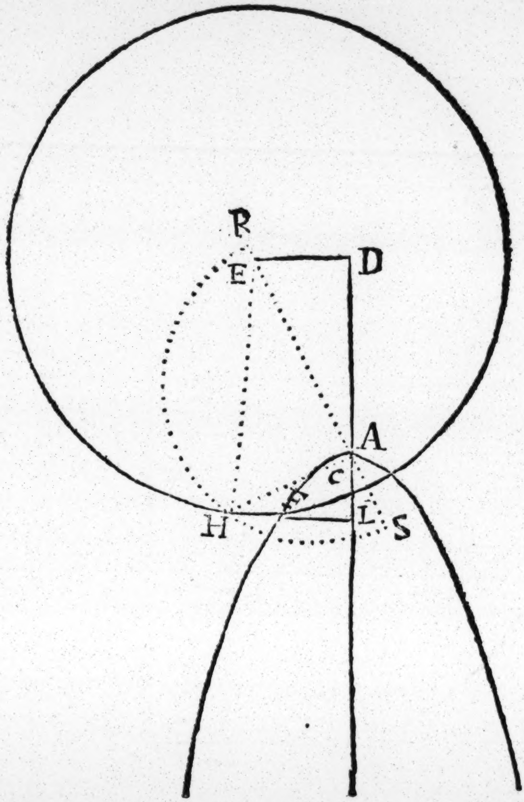


Radius of the Circle that is to cut the Parabola. (*See also Fig. following.* Where the Point D, is in the Axis produced above the Vertex.)

But if the Quantity r happen to have a Negative Sign in the Equation; there must yet another Circle be described on AE as a Diameter: In which accommodate AI = to AH the Perpendicular before found; which will find the Point I, through which the Intersecting Circle must pass: And whose Radius will be IE, and Center E, as before.

See Fig. in Pag. 84.

And

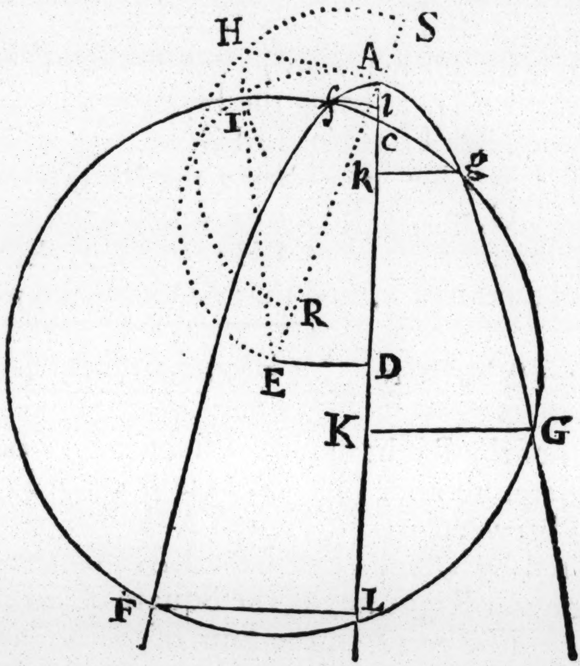


And by this means a Circle will be drawn, which will cut the Parabola in 1, 2, 3 or 4 Points, from whence Perpendiculars let fall to the Axis, will be all the possible or real Roots of the Equations, Affirmative or Negative: The former of which will be on the side of the Axis that the Center E is, when 'tis $+ q$, but on the other Side, when it is $- q$.

The Demonstration of all which he thus very easily gives us.

Call

Call the Ordinate G K, by this Conſtruction found to be a true Root, by the Name of z: Then 'tis plain the Abſciſſa, A K muſt be zz, becauſe in the Parabola the Ordinate is always a mean Proportional between the Parameter (here ſuppoſe



$a = 1$) and the Abſciſſa; by the firſt Property of this Figure above demonſtrated. Wherefore if from AK you take A C $= \frac{1}{2}$ (or $\frac{1}{2}a$) and then C D $= \frac{1}{2}p$: The Remainder DK (=EM) will in this Notation be $zz - \frac{1}{2}p - \frac{1}{2}$. (*See the two former Figures.*) And this Quantity $zz - \frac{1}{2}p - \frac{1}{2}$ ſquared, produces $z^4 - zzp - zz + \frac{pp}{4} + \frac{p}{2} + \frac{1}{4}$. And becauſe by the Conſtruction D E

or KM $= \frac{1}{2}q$, therefore the whole GM will be in this Notation $= z + \frac{1}{2}q$, whose Square is $zz + zq + \frac{qq}{4}$. Add then this and the former Square of DK or EM together, and it makes $z^4 - zzp + qz + \frac{1}{4}pp + \frac{1}{4}qq + \frac{1}{2}p + \frac{1}{4}$; which (by 47. *e.* 1. *Eucl.*) will be equal to the Square of EG, as being the Hypothenuse of the Right-angled Triangle EMG.

But this Line EG, being the Radius of the Circle FG, may easily be expressed another way, by taking for it its equal EH: For since ED was taken equal to $\frac{1}{2}q$, and that AD was $= \frac{1}{2}p + \frac{1}{2}$, EA must be = to $\sqrt{: \frac{qq}{4} + \frac{pp}{4} + \frac{p}{2} + \frac{1}{4}}$, by reason of the Right-angled Triangle ADE.

Wherefore also, Since AH is a mean Proportional between AS = 1, and AR $= r$, it must be noted by $\sqrt{: r}$. And since EAH by the Supposition is a Right-angle, the Square of EH (*i.e.* EG) will be equal to the Sum of the Squares of HA and of EA; Now the Square of EA is $\frac{qq}{4} + \frac{pp}{4} + \frac{p}{2} + \frac{1}{4}$: To which adding the Square of AH $= r$, it will stand thus, $\square$ EH $= \frac{1}{4}qq + \frac{1}{4}pp + \frac{1}{2}p + \frac{1}{4} + r$: And since EG = EH, these two Quantities will be equal, *viz.* $z^4 - zzp + qz + \frac{1}{4}pp + \frac{1}{4}qq + \frac{1}{2}p + \frac{1}{4} = \frac{qq}{4} + \frac{pp}{4} + \frac{p}{2} + \frac{1}{4} + r$: Compare then these two and

and reject what is common to both; and you will find remain $z^4 + pzz - bz + r = 0$.

From whence it appears, That the Ordinate G K, which we called z, is the true Root of the Equation propoſed to be conſtructed. *Q. E. D.*

And if you apply this Calculation to all other Caſes of this Rule, changing the Sign + or —, as occaſion requires, you will gain your Deſign after the ſame manner.

Thus far went *Des Cartes* in this Matter ; but he conſidering only the Axis of the Parabola, and not thinking what might be done by the other Diameters, or Parallels to the Axis, could not this way conſtruct either *Cubick* or *Biquadratick Equations*, till he had firſt ejected or taken away the ſecond Term : Which to effect is a tedious and troubleſome Operation.

All which our Famous Mr. *Thomas Baker*, Rector of *Nympton*, in the County of *Devon*, well conſidering, and having withal ſeen how near *Schooten* was of gaining the Point, (who drew a Parallel to the Axis without the *Curve*, and by that means conſtructed *Cubicks* without taking away the ſecond Term. See his Comment on *Des Cartes*'s 3d Book, Page 328) He thought of drawing a Parallel to the Axis within the Figure ; whereby, and by the help of Mr. *Strode*'s Property above mentioned, viz. *That the Parameter is to the Sum of any two Ordinates* : : *as their Difference is to the Difference of their Abſciſſa* : He found he could conſtruct all ſorts of Equations, not exceeding four Dimenſions ; whether all their Terms were there, or whether the ſecond, third, or fourth Term, or all of them were wanting. And that he could find the Center of a Circle which would

cut the Parabola in as many Points as the Equation had real Roots: which real Roots, would now be Perpendiculars let fall from those intersected Points of the *Curve*, to the said Diameter or Parallel to the Axis. Of this he gives abundance of Examples and Cases, with their Demonstrations, in his *Geometrical Key*, or *Gate of Equations unlock'd*. And because the chief, or rather only Difficulty, lies in finding the Center of a Circle which shall Intersect the Parabola in the Points required: He gives us for this purpose, what he very properly calls his *Central Rule*, which is a Rule consisting of two Parts: By the former of which he determines the Point D in the Parallel to the Axis, from whence the Perpendicular DE is to be erected: And by the latter, the Length of that Perpendicular, whereby he finds the Point E, the Center of the Circle required. *See the following Figure in Pag.* 89.

His Central Rules are these.

$$\text{I. } \frac{L}{2} + \frac{pp}{8L} \pm \frac{q}{2L} = b = AD.$$

$$\text{II. } \frac{p}{4} + \frac{ppp}{16LL} \pm \frac{q}{4LL} \pm \frac{r}{2LL} = d = DE.$$

And because L the *Latus Rectum* or *Parameter* of the Parabola is equal to 1, it may be contracted in this Form.

I.

$$\text{I. } \frac{1}{2}+\frac{pp}{8}\pm\frac{q}{2}=b=AD.$$

$$\text{II. } \frac{p}{4}+\frac{ppp}{16}+\frac{pq}{4}+\frac{r}{2}=d=ED.$$

The Analytical Inveſtigation of which Central Rule Mr. *Baker* gives, tho' obſcurely, in his above mentioned Book. And 'tis done alſo much clearer by *Sturmius* in the Appendix to his Introduction to his *Specioſa Analyſis*, at the end of his *Matheſis Enucleata*, which the Reader would do well to conſult.

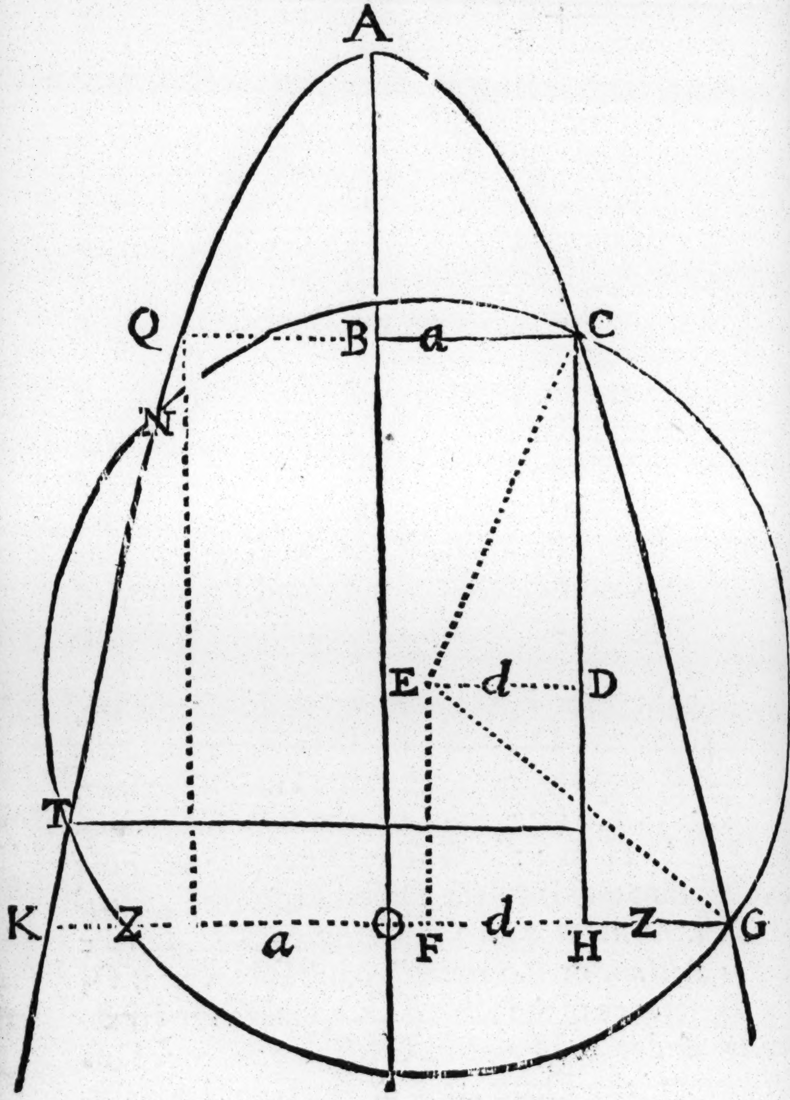
The Invention, Reaſon, and Demonſtration of which certain Rule is very briefly and clearly ſhown, as follows; in which I was aſſiſted by that learned Algebraiſt Mr. *Abraham de Moivre*.

Suppoſe any Parabola drawn, as KAG. This Lemma which expreſſes the ſecond Property of that Figure, as above mentioned, may be premiſed.

That the Line KG being an entire Ordinate, and cutting the Diameter or Parallel to the Axis CH, at Right-angles in the Point H. 'Tis plain, that L : HK (the Sum of the two Ordinates GO + OH as : : HG (the Difference of thoſe two Ordinates) is to the Difference of the *Abſciſſa* CH. Wherefore the Rectangle under L, (the *Latus Rectum* or *Parameter*) and CH (= BO) the Difference of the *Abſciſſa*, is equal to the Rectangle under KH and HG (the Sum and Difference of the Ordinates) or equal to the Rectangle KHG.

Suppoſe

Suppose then in the Figure annexed, Mr. *Baker's* L, the Parameter, or *Latus Rectum* belonging to this Parabola, to be called $p = 1$.



Let $CB = a$. $GH = z$ the Root sought; then will be
$CD = b$. $FG = z + d$. And
$DE = d$. $HK = 2a + z$.

Wherefore by the Lemma, or second Property of the Parabola $p \times$ by $CH = 2az + zz$.

Wherefore $CH = \frac{2az + zz}{p}$, and EF $(= DH) = \frac{2az + zz}{p} - b$; but (by 47. e.1. *Eucl.*) $FEq, + FGq; = EGq; = ECq = EDq + DCq$. That is, in this way of Notation,

$$\frac{4aazz + 4az^3 + z^4}{pp} - \frac{4baz - 2bzz}{p} + bb + zz + 2zd + dd = bb + dd.$$

Strike out $bb + dd$ being common to both Sides of the Equation, and it will stand thus,

$$\frac{4aazz + 4az^3 + z^4}{pp} - \frac{4baz - 2bzz}{p} + zz + 2dz = 0.$$

After this, if you multiply the last Equation by pp, and then divide the Product by z, you will bring it to this Form, $4aaz + 4azz + zzz - 4pba - 2pbz + ppz + 2ppd = 0$.

Or, To reduce it to a more regular Form, where the Terms shall be ranged in their proper Order: Let it stand thus.

$$\begin{aligned} Z^3 + 4azz + ppz &- 4bpa \\ + 4aaz &+ 2ppd = 0 \\ - 2bpz. & \end{aligned}$$

Or, If you had called the whole Line C Q, by the Name of a, as *Baker* calls it p, the Equation would ſtand thus.

$$\begin{aligned} Z^3 + 2aaZZ + ppz &- 2bpa \\ + aaz &+ 2pbd = 0 \\ - 2bpz. & \end{aligned}$$

And this is a Cubick Equation produced, having all its Parodick Degrees, or Terms, and one of whoſe true Roots is apparently z, a Right-line let fall from G, the Point of Interſection of the Circle and the Parabola, to the Diameter or Parallel to the Axis C H.

Now if you will compare this Equation, Member by Member, with one given to be ſolved in the common Form, as ſuppoſe $Z^3 + m\,ZZ + rZ - S = 0$, you will find all things reſpectively equal. Deſcribe any *Parabola*, as K A G in the laſt Figure, where apply Q C at Right-angles to the Axis $= \frac{1}{2}m$. Then compare the *Co-efficients* in the two next correſponding Terms in both Equations, and you will have $pp + aa - 2pb$ in the former, $= r$ in the latter or common Equation. This is (becauſe $2a$ the Co-efficient of the ſecond Term in one, is equal to m in the other Equation, and conſequently $a = \frac{1}{2}m$) $pp + \frac{1}{4}mm - 2pb = r$.

Where-

Where $b = \frac{pp + \frac{1}{4}mm - r}{2p} = \frac{1}{2}p + \frac{mm}{8p}$ $+ \frac{\frac{1}{2}r}{p}$: Which is the Rule to find the length of CD, or to determine the Point D, nearly the ſame, changing only the Letters with Mr. *Baker*'s firſt Rule, *viz.* $\frac{L}{2} + \frac{pp}{8L} + \frac{q}{2L} = b = AD$.

Then again, compare the next two correſponding and equal Terms in both Equations together, and you will have $2pbd - 2bpa = S$; wherefore $d = \frac{S + 2bpa}{2pb}$; which being all a known Quantity, the length of the Line E D is known; and conſequently the Central Point E is found, on which the Circle is to be deſcribed with the Radius E C, or E G.

For in every *Cubick Equation*, where are all the Terms, the Circle will paſs thro' the Point C, the Vertex of the Diameter or Parallel to the Axis. And the Circle will cut or touch the Parabola in as many Points as the Equation hath *real Roots*, which will be *Perpendiculars*, let fall from thoſe Points to the Diameter C H. And which of them are Poſitive or Affirmative, and which are Negative, hath been ſhewn above.

But if the Equation had been a *Biquadratick*, the Circle will not paſs thro' the Vertex of the Diameter, but thro' another Point, to be found according to the Rule above mentioned in *Des Cartes* his Conſtruction, which is to ſet

$\sqrt{\frac{S}{pp}}$ (if S represent the Absolute Number or fifth Term in such an Equation, and have a Negative Sign) at Right-angles to E C, on the Vertex of the Diameter; but if it be + S, then that Line $\sqrt{\frac{S}{pp}}$ must be inscribed in another Semicircle made on the Line E C; which being set from the Vertex of the Diameter C, will give in the Periphery of that Semicircle on E C, a Point thro' which the Intersecting Circle required should pass.

And thus are *Cubick* and *Biquadratick Equations* constructed, having all their Terms. But such as want the second, third, or any other Term or Terms, may be as well constructed this way, only by leaving out of the Central Rule that Part of it which belongs to such *wanting Terms*, and going on as is above shew'd with the rest.

If you would see how the Central Rule is Investigated in the last Term of a *Biquadratick*, (as well as here in a Cubick) according to *Baker's* Method.

Suppose QC as before $= a$. Then if z in the Figure be taken for one of the real Roots of a *Biquadratick*, a regular Equation will be produc'd in this Form.

$$\left.\begin{array}{llll} Z^4. & 2az^3. & aazz. & 2pabz \\ & & 2pbzz. & 2pbdz - S \\ & & ppzz. & \end{array}\right\} = 0.$$

Compare

Compare this as is above ſhew'd, with any perfect *Biquadratick* in the Common Form: As ſuppoſe, $Z^4.\ m z^3.\ q z z.\ r z.\ S = 0$. Firſt, make $QC = \frac{1}{2} m$; which is equal every where to a in the other Equation, and putting therefore half m inſtead of a, that will ſtand thus.

$$\left.\begin{array}{l} Z^4 + m z^3 + \frac{mm}{4} z z - p m b z - S \\ \qquad - 2 p b z z + 2 p b d z \\ \qquad + p p z z \end{array}\right\} = 0.$$

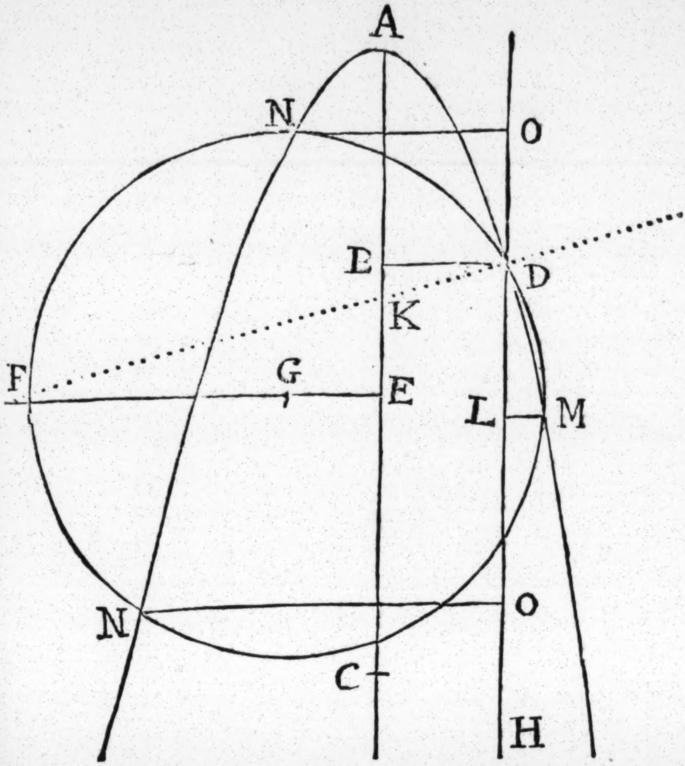
The ſecond Term being expreſs'd by QC, go on to compare the third Term in each, ſo you will have $\frac{mm}{4} - 2 p b + p p = q$. Wherefore, $p p + \frac{mm}{4} - q = 2 p b$. And conſequently dividing all by $2 p$ $\frac{p}{2} + \frac{mm}{8 p} - \frac{q}{2 p} = b = CD$, which is the firſt part of the Central Rule.

And ſince $- p m b + 2 p b d = r$, therefore $p m b + r = 2 p b d$. But $b = \frac{p}{2} + \frac{mm}{8 p} - \frac{q}{2p}$: Wherefore ſubſtituting this inſtead of b in the laſt Equation; it will be $\frac{p p m}{2} + \frac{m m m}{8} + \frac{m q}{2} + r = 2 p b d$: And conſequently dividing all by $2 p b$, it

it will stand thus; $\frac{m}{4} + \frac{m^3}{16pp} + \frac{mq}{4pp} + \frac{r}{2pp}$ $= d =$ D E; which is the second Part of the Central Rule in a Compleat *Biquadratick*.

That Excellent Mathematician, Dr. *Edmund Halley*, in his *Philosophical Transactions*, N. 188. hath a peculiar Dissertation on this Subject: *Of the Construction of Solid Problems*; or *Of Equations of the third or fourth Power*. In which he not only gives the Reason and Foundation of Mr. *Baker*'s Rule, gets rid of the Intricate Cautions of *Baker*, in reference to the Signs, *&c.* but he gives also a new Construction of those Equations which is very easie and short: And which therefore I shall now annex to what hath been already done.

Let any *Cubick* or *Biquadratick Equation*, having all its Terms, be given to be constructed in one of these Forms, $Z^3.\ bzz.\ apz.\ aaq = 0.$ or $Z^4.\ bzzz.\ apzz\ \ aapz.\ a^3.\ r = 0.$ to which it will be capable of being reduced. Then describe a *Parabola*, as N A M, whose *Parameter* or *Latus Rectum* let be a: Its Vertex A, and its Axis A B C. Then apply at Right-angles to the Axis B D $= \frac{1}{4} b$, the second Term in the Equation; and thro' D draw DH parallel to the Axis, and let it be placed on the Left-hand, if b have a Negative Sign; but on the Right-hand if it be $+ b$. In the Line A B *continued* downwards towards B, take BK $= \frac{1}{2} a$: and then draw the Infinite Line F K D, take K C $=$ 2 A B, always downwards from K; and if the Quantity p have a Negative Sign, take also the same way C E $=$ $\frac{1}{2} p$; but on the contrary, take it upwards if it be



be $+ p$; then at the Point E thus found (or from the Point C, if q be *wanting*) let there be erected E F, perpendicular to the Axis, which ſhall cut the Line D K, when produc'd in the Point F. Which Point F, if the Quantity q be wanting, ſhall be the Center of the Circle required; but if q be in the Equation, and have a Negative Sign, then ſet towards the Right-hand F G $= \frac{1}{2} q$; but if q have a Poſitive Sign, F G muſt be ſet on the Left-hand of F, on E F produced that way. So ſhall G be the Center of the Circle required, and G D its Radius, if the Equation be a *Cubick*. But if it be a *Biquadratick*, then the Square of C D muſt

must be augmented if it be $-r$, or diminished if it be $+r$ by the Addition or Subtraction of the Rectangle under r, and the Parameter (which is very easie, as hath been before shew'd, to effect Geometrically) Then a Circle described with the Radius thus encreased, shall intersect the Parabola in as many Points as the Equation hath real Roots; and Perpendiculars from those Points let fall to the Diameter D H, shall be those real Roots in the Equation proposed: Whereof the Affirmative ones M L will be on the Right-hand, and the Negative ones N O on the Left.

And much after the same manner doth he shew us how to construct *Cubick Equations*, (having all their Terms) according to *Schooten*'s Rule, whereby the Roots are refer'd to the Axis. And because *Schooten* neither gives the Invention nor Demonstration of this Rule, Dr. *Halley* shews its Original to be this: That every *Cubick Equation*, having all its Terms, may be reduced to a *Biquadratick* (where the second Term is wanting) by multiplying such *Cubick Equation* by $z - b = o$, if it be $+b$, or $Z + b = o$, if it be $-b$. Which new produced Equation shall have the same Roots as the *Cubick* had, and also one more equal to $\mp b$, according as the Sign of b was in the Equation. As for Instance, Let this Equation $Z^3 - zzb + apz - aaq = o$, be proposed to be constructed.

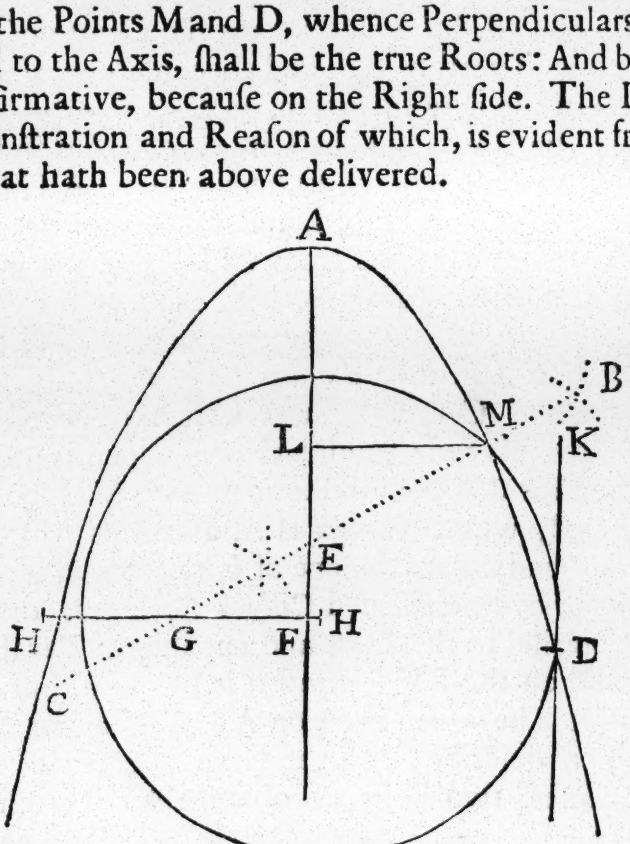
This multiplied by $z + b$ (because b hath a Negative Sign) makes $Z^4 - z^3 b + apzz + aaqz + z^3 b - zzbb + apzb + aaqb$, which Equation, when considered, will want its second Term, because $-Z^3 b$ and $+z^3 b$, do destroy one another. So it will stand thus,

$$Z^4 * + apzz + aaqz$$
$$- bbzz + abzp + aaqb = 0.$$

Where the Co-efficients of the third Term, *viz.* $-bb + ap$, do give $-\frac{bb}{2a} + \frac{1}{2}p$, to be taken instead of $\frac{1}{2}p$, or C D, in *Des Cartes* Construction: And the Co-efficients of the fourth Term, $aaq + abp$ give $\frac{1}{2}q + \frac{bp}{2a}$ to be taken instead of $\frac{1}{2}q$ or D E, by which means the Center of the Circle is determined; and because one of the Roots of the new Equation, *viz.* $+b$, or $-b$ is given, the Radius of the Circle, or Point in the Circumference will be known also. A Circle then being thus described, and Perpendiculars being let fall to the Axis from its Intersection with the *Curve* of the *Parabola*, they shall be the true Roots required, the Negative ones on the Left-hand, and Affirmative ones on the Right.

And the Center of the Circle required is found by this easie Construction; which is much the best for *Cubicks*. A *Parabola* as AMD being described, whose Vertex is A, and Axis AH: At the Perpendicular distance of *b*, the second Term in the above mentioned *Cubick Equation*, draw KD Parallel to the Axis, and cutting the Parabola in D, on the Right-hand if it be $+b$, but on the Left it be $-b$. Then on A and D, as on two Centers, describe with the same opening of the Compasses, two Pairs of obscure Arks crossing each

each other, as you see in the following Figure, and thro' which is to be drawn the infinite Line B C at Right Angles with the supposed Line AD, and cutting the Axis in the Point E. Then set (downward from E if it be $-p$, but upward towards A if it be $+p$) $EF = \frac{1}{2}p$. And from F (or from E if p be wanting) erect the Perpendicular Line F G, cutting the infinite one B C, in the Point G. Lastly, Produce $GH = \frac{1}{2}q$ (towards the Right-hand if it be $-q$, but towards the Left-hand, if it be $+q$) and then shall H be the Center of the Circle required, and HD the Radius. Which Circle shall intersect the Parabola in the Points M and D, whence Perpendiculars let fall to the Axis, shall be the true Roots: And both Affirmative, because on the Right side. The Demonstration and Reason of which, is evident from what hath been above delivered.



The

The learned Dr. *Halley* in *Philoſ. Tranſactions*, N. 190. hath an excellent Diſcourſe of the Number of Poſſible Roots in a *Solid*, or *Biquadratick Equation*; and alſo of their Limits: Where he ſhews, from the Principles of the above written Method of Conſtructions, That ſince a Circle interſecting a Parabola muſt do it either in two Points or four: Therefore in *Biquadratick Equations*, there will be either two or four real Roots, Affirmative or Negative. And that if it happens that the Circle touch the Parabola only in any Point, and do not cut it: 'Tis then an Indication of the Equality of two Roots having the ſame Sign.

But in *Cubicks* of all ſorts, and however *Adfected*, there is either but one, or elſe three Poſſible Roots, ſuppoſing you allow Negative ones as ſuch.

In *Biquadraticks*, if the laſt Term *r* have a Negative Sign, there are always either two or four Roots. But if it be $+ r$, and alſo that $\sqrt{: GD^q - ar}$, *(See the laſt Figure ſave one)* be ſo ſmall, that the Circle deſcribed with that Radius on the Center G, cannot touch or cut the Parabola in any Point: Then is that Equation utterly impoſſible; and is explicable by no poſſible Root; either Affirmative or Negative. By what means he attains to the Knowledge of theſe Rules and Limitations, the Reader may find there at large, where he fully illuſtrates all things with proper Examples.

OF

SURD ROOTS.

WHEN any Number or Quantity hath its Root proposed to be extracted, and yet is not a *true Figurate Number of that kind*: That is, if its Square Root being demanded, it is not a *true Square*: If its Cube Root being requir'd, it self be not a *true Cube*, &c. then 'tis impossible to assign, either in whole Numbers or Fractions, any exact Root of such Number proposed. And whenever this happens, 'tis usual in *Mathematicks* to mark the required Roots of such Numbers or Quantities, by prefixing before it the proper Mark of Radicality, which is $\sqrt{\ }$: Thus $\sqrt{\ }$: 2, signifies the Square Root of 2, and $\sqrt[3]{\ }$: 16, or $\sqrt{\ }$: (3) 16, signifies the *Cubick Root* of 16: Which Roots, because they are impossible to be expressed in Numbers exactly (for no effable Number, either Integer or Fraction multiplied into it self can ever produce 2; or being multiplied Cubically can ever produce 16) are very properly call'd *Surd Roots*.

There is also another way of *Notation* now much in Use, whereby Roots are expressed without the Radical Sign, by their *Indexes*: Thus, as x^2, x^3, x^5,

x^5, ſignifie the Square, Cube and 5th Power of x; ſo $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{5}}$, *&c.* ſignifie the Square Root, Cube Root *&c.* of x. The Reaſon of which is plain enough, for ſince $\sqrt{}$: is a Geometrical mean Proportional between 1 and x: So half is an Arithmetical mean Proportional between 0 and 1, and therefore as 2 is the Index of the Square of x, half will be the proper Index of its Square Root, *&c.*

Obſerve alſo, that for Convenience or Brevity ſake, Quantities or Numbers which are not *Surds*, are often expreſſed in the Form of *Surd Roots*: Thus, $\sqrt{}:4$, $\sqrt{}:\frac{9}{4}$, $\sqrt[3]{}:27$, *&c.* ſignifie 2, $\frac{2}{3}$, 3, *&c.*

But altho' theſe *Surd Roots* (when truly ſuch) are inexpreſſible in Numbers, they are yet capable of Arithmetical Operations (ſuch as *Addition*, *Subtraction*, *Multiplication*, *Diviſion*, &c.) which how readily to perform the Algebraiſt ought not to be Ignorant.

Surds are either *Simple*, which are expreſſed by one ſingle Term; or elſe *Compound*, which are formed by the Addition or Subtraction of Simple Surds: As $\sqrt{}:5 + \sqrt{}:2$ $\sqrt{}:5 - \sqrt{}:2$, or $\sqrt[3]{}:\overline{7 + \sqrt{}:2}$: Which laſt is called an *Univerſal Root*: And ſignifies the *Cubick Root* of that *Number which is the Reſult of adding* 7 *to the Square Root of* 2.

The Arithmetick of Surds consists of these principal Parts.

I. *To reduce Rational Quantities to the Form of any Surd Roots assigned.*

Which is perform'd by involving the Rational Quantity according to the Index of the Power of the Surd, and then prefixing before it the Radical Sign of the Surd proposed.

Thus to reduce $a = 10$ to the Form of $\sqrt{}: 15, = \sqrt{}: b$, you must square $a = 10$; and prefixing the Sign, it will stand thus, $\sqrt{}: aa = \sqrt{}: 100$, which is in the Form of the Surd desired. So also if 3 were to be brought to the Form of $\sqrt[4]{}: 12$, you must raise 3 up to its fourth Power, and then prefixing the Note of Radicality to it, it will be $\sqrt[4]{}: 81$, or $81^{\frac{1}{4}}$, which is the same Form with $\sqrt[4]{}: 12$.

And this way may a Simple Surd Fraction, whose Radical Sign refers only to one of its Terms, be changed into another which shall respect both Numerator and Denominator. Thus, $\frac{\sqrt{}: 2}{5}$ is reduced to $\sqrt{}: \frac{2}{25}$ and $\frac{5}{\sqrt[3]{}: 4}$ to $\sqrt[3]{}: \frac{125}{4}$ where the Radical Sign affects both Numerator and Denominator alike.

II. *To*

II. *To reduce Simple Surds, having different Radical Signs* (which are called *Heterogeneal Surds*;) *to others that may have one common Radical Sign, or which are Homogeneal.*

Divide the Index of the Powers of the Surds by their greateſt common Diviſor, and ſet the Quotients under the Dividends; then multiply thoſe Indexes croſs-ways by each others Quotients, and before the Products ſet the common Radical Sign $\sqrt{}$: with its proper Index: Then involve the Powers of the given Roots alternately, according to the Index of each others Quotient, and before thoſe Products prefix the common Radical Sign before found.

To reduce $\sqrt[2]{}:aa$ and $\sqrt[4]{}:bb$.

$2)\ \sqrt{}:aa \qquad 2)\ \sqrt[4]{}:bb$

$1 \quad \mathsf{X} \quad 2$

$\sqrt[4]{}:bb \qquad \sqrt[4]{}:aaaa.$

To reduce $\sqrt{}:5$ and $\sqrt[4]{}:7$.

$\sqrt[2]{}\,5 \qquad \sqrt[4]{}:7$

$1 \quad \mathsf{X} \quad 2$

$\sqrt[4]{}:25 \qquad \sqrt[4]{}\,2401.$

III. To

III. *To reduce Surds to their lowest Terms possible.*

Divide the Surd by the greatest Square, Cube, Biquadrate, *&c.* or any other higher Power, which you can discover is contained in it, and will measure it without any Remainder, and then prefix the Root of that Power before the Quotient, or Surd so divided, and this will produce a new Surd of the same Value with the former, but in more simple Terms. Thus, $\surd : 16\,aab$, by dividing by $16\,aa$ and prefixing the Root $4\,a$, will be reduced to this $4\,a\surd : b$ and $\surd : 12$ will be depress'd to $2\,\surd : 3$. Also $\sqrt[3]{} : cb^3r$, will be brought down to $b\sqrt[3]{} : cr$. And this Reduction is of great Use whenever it can be perform'd: But if no such Square, Cube, Biquadrate, *&c.* can be found for a Divisor; then you must find out all the Divisors of the Power of the Surd propos'd; and then see whether any of them be a Square, Cube, *&c.* or such a Power as the Radical Sign denotes; and if any such can be found, let that be used in the same manner as is above said, to free the Surd Quantity in part from the Radical Sign. Thus, If $\surd : 288$ be propos'd; among its Divisors will be found the Squares 4, 9, 16, 36, and 144, by which if 288 be divided, there will arise the Quotients 72, 32, 18, 8 and 2, wherefore instead of $\surd : 288$, you may put $2\,\surd : 72$, or $3\,\surd : 32$, or $4\,\surd : 18$, or $6\,\surd : 8$, or lastly, $12\,\surd : 2$, and the same may be done in Species.

IV. *To*

IV. *To find whether two Surd Roots given are Commensurable or not.*

Thoſe are called *Commenſurab'e Surds*, which are to one another as Number to Number, as one Rational Quantity to another; or which are, when reduced to their leaſt Terms, true Figurate Quantities of their own kind.

To diſcover therefore, whether they are ſuch or not; If the Surds are of different kinds (or *Heterogeneal Surds* as ſome call them) they muſt firſt be reduced to one kind, and then divided ſeverally by their greateſt common Meaſure, for if then there will come out Rational Quotients, the firſt Surds are *Commenſurable*; but if the Quotients are Irrational, or Surd Numbers or Quantities, then the propoſed Surds are Incommenſurable.

V. gr. To examine whether $\sqrt{}:12$, and $\sqrt{}:3$, are Commenſurable Surds? They being Homogeneal, I divide them ſeverally by their greateſt Common Diviſor, which is $\sqrt{}:3$; and the Quotients are $\sqrt{}:4$, and $\sqrt{}:1$, that is, 2 and 1. Wherefore, ſince 2 and 1 are Rational Numbers, I ſay that $\sqrt{}:12$ and $\sqrt{}:3$, are Commenſurable Surds; or are to one another, as 2 to 1, which is very plain; for no doubt 12 : 3 : : as 4 : 1, and 'tis plain, that as Squares are to one another, ſo are their Roots: Wherefore 12 : 3, as $\sqrt{}:12$, $\sqrt{}:3$, that is, as $\sqrt{}:4$, $\sqrt{}:1$, or as 2 to 1.

Whenever two Surds are divided by one common Diviſor, (tho' not the greateſt) if their Quotients come out Rational, or are to one another as Number to Number, thoſe Surds are certainly Commenſurable.

If

If Fractional Surds were given, not having a common Denominator, they muſt firſt be reduced to their ſmalleſt common Denominator, and then if their Numerators are commenſurable, you may conclude the firſt Surd Fractions were ſo.

But if either the Numerators or Denominators of two Surds proper Fractions, or mixt Numbers in the Form of Fractions (neglecting the Radical Sign) be Powers of that kind which the Radical Sign expreſſes, then they will need no Reduction: For if their Numerators or Denominators are Commenſurable, the whole Surd Fractions propoſed are certainly ſo. Thus, If it were enquired, Whether $\sqrt{::\frac{50}{16}}$ and $\sqrt{:\frac{72}{25}}$ are Commenſurable Surds; becauſe 16 and 25 are Squares, or ſuch Powers as the Radical Sign expreſſes or denotes, omitting the Sign $\sqrt{}$: you need only compare the Numerators $\sqrt{:50}$, and $\sqrt{:72}$; which being divided by their greateſt common Diviſor $\sqrt{2}$, the Quotients will be 5 and 6 (*i. e.* $\sqrt{:25}$ and $\sqrt{:36}$) Wherefore the given Surds are Commenſurable, and are to one another, as $\frac{5}{4}$ to $\frac{6}{5}$; and conſequently, by the precedent Rule, may be expreſſed thus, $\frac{5}{4}\sqrt{:2}$ and $\frac{6}{5}\sqrt{:2}$.

For an Inſtance in *Species*: Suppoſe that it were enquired whether $\sqrt{:27aa}$, and $\sqrt{:12aa}$ were Commenſurable Surds? Divide each by the greateſt common Diviſor, $\sqrt{:3aa}$: And the Quotients $\sqrt{:9}$ and $\sqrt{:4}$; that is, 3 and 2 are Rational Numbers; and conſequently, the propoſed Surds are Commenſurable.

Multiplication

Multiplication of Simple Surd Roots.

IF the Surds propoſed be of the ſame kind, multiply them one by another, and prefix the common Radical Sign to the Product; but if the Surds are Heterogeneal, or of different kinds, they muſt be reduced firſt (according to *Rule* 2.) to Surds having the ſame Radical Sign.

Thus to multiply $\sqrt{:}\,7$ by $\sqrt{:}\,8$. the Product will be $\sqrt{:}\,56$.

For ſince in all Multiplication, as 1 is to one Factor, ſo is the other to the Product; therefore here $\sqrt{:}\,1 : \sqrt{:}\,7 :: \sqrt{:}\,8 : \sqrt{:}\,56$. Wherefore $1 : 7 :: 8 : 56$, that is, 56 is the true Square of $\sqrt{:}\,56$, and $\sqrt{:}\,56$, the true Root of $7 \times 8 = 56$.

Other Examples.

I. If $\sqrt{:}\,8$ were to be multiplied into $\sqrt[3]{:}\,4$, becauſe they are not Homogeneal Surds, they muſt be reduced to ſuch by *Rule* 2, and then they will ſtand thus; $\sqrt[6]{:}\,512$ $\sqrt[6]{:}\,16$, which being multiplied into each other, and the common Radical Sign prefix'd, will make $\sqrt[6]{:}\,8192$: And thus the $\sqrt[6]{:}\,27$ multiplied by $\sqrt{:}\,9$, when reduced, and rightly multiplied, produces $\sqrt[6]{:}\,531441$.

 II. When

II. When a Surd is to be multiplied by a Rational Quantity, that Rational Quantity ought first to be reduced to a Surd of like Nature with the true Surd. But 'tis oftentimes convenient only to connect them together, by prefixing the Rational Quantity to the Left-hand of the Surd. As suppose $\sqrt{:27}$ were to be multiplied by 6, the Product may commodiously be expressed thus, $6\sqrt{:27}$, and so if $\sqrt[4]{:9}$ were to be multiplied by 10, it will stand thus, $10\sqrt[4]{:9}$.

III. And when two Rational Quantities are thus prefix'd to two Surds of the same kind, you may find the Product of them by multiplying the Rational Part by the Rational, and the Surd Part by the Surd, then those joyned together will be the Product required. Thus $6\sqrt[3]{:7}$ multiplied by $5\sqrt[3]{:}$ produces $30\sqrt[3]{:21}$.

IV. If any Surd Root be to be multiplied into it self or *Involved*, according to the Index of its proper Power, you need only cast away the Radical Sign, and then the Quantity or Number remaining is always the Square, Cube, or other Power required; and will always be Rational. Thus the Square of $\sqrt{:11}$ is 11. The Cube of $\sqrt[3]{:30}$, is 30; also $2\sqrt{:3}$ multiplied by $8\sqrt{:3} = 48$, and $3\sqrt{:5}$, multiplied by $2\sqrt{:5} = 30$.

V. And

V. And if the Index of the Power be any even compound Number greater than two, and 'tis required to ſquare ſuch a Surd : There need only a Radical Sign, whoſe Index is half the former, be prefix'd to the Quantity, inſtead of the former Compound one, and it is done. *V. gr.* Suppoſe you would *Square* this Surd, $\sqrt[4]{}$: 12 ; becauſe the Index 4 is compounded of 2 and 2 ; $\sqrt{}$: 12 is the true Product, or the true Square of the Surd Root $\sqrt[4]{}$: 12, ſo alſo the Square of $\sqrt[6]{}$: 10, is $\sqrt[3]{}$: 10.

But when a Simple Surd Quantity, whoſe Radical Sign hath for its Index ſome Ternary Number greater than 3, as 6, 9, &c. And 'tis required to involve this Surd Cubically. Then only prefix before the Quantity a Radical Sign with an Index, which is one third of the former, and 'tis done. Thus, If $\sqrt[6]{}$: 64 were to be Cubed, it will be $\sqrt[3]{}$: 64, and the Cube of $\sqrt[9]{}$: 512, is $\sqrt[3]{}$: 512, &c. alſo the Biquadrate of $\sqrt{}$: 5, is 25 (as being the Square of the Square of $\sqrt{}$: 5) And the Cube of $\sqrt[6]{}$: 81 will be $\sqrt{}$: 81 or 9.

In the general, to *Square*, *Cube*, &c. any Surd Root, is only to Square or Cube the Power, retaining the ſame Note of Radicality ; but 'tis better where it can be done, to take one half, one third part, &c. of the Exponent of the Root, as is above ſhewn in the laſt particular Rules (On the con-

trary, If you would extract the *Square*, *Cube*, or other Root of any Surd, you must double or triple, *&c.* the Exponent of the Radicality. Thus the Square Root of $\sqrt{\ }:16$, is $\sqrt[4]{\ }:16$, the Square Root of $\sqrt[3]{\ }:27$, is $\sqrt[6]{\ }:27$, *&c.*

Division of Simple Surd Roots.

I. IF the Surds are Similar, *Homogeneal*, or of *the same kind*, divide one Number or Quantity by another, and prefix the common Radical Sign to the Quotient: But if they are Heterogeneal, or not of the same kind, they must be *reduced* before they can be divided. Thus, $\sqrt{\ }:9.)\ \sqrt{\ }:576\ (\sqrt{\ }:64 = 8$. And $\sqrt[4]{\ }:5\ (\sqrt[4]{\ }:35\ (\sqrt[4]{\ }:7$.

The Demonstration of which General Rule is the same as that in Multiplication; for from the Nature of Division, the Divisor is to Unity :: as the Dividend to the Quotient. Therefore in our first Instance, $\sqrt{\ }:9 : \sqrt{\ }:1 :: \sqrt{\ }:576 : \sqrt{\ }64$, but as these Roots are, so will their Squares be: That is, $9 : 1 :: 576 : 64$, and that these Numbers are truly Proportional, is apparent; because the Rectangles of the Extreams and Means are equal. Wherefore, $\sqrt{\ }:9, \sqrt{\ }:1 :: \sqrt{\ }:576, \sqrt{\ }:64$, and consequently 64 is the true Quotient.

II. If

II. If any Rational Quantity be to be divided by its Square Root, the Square Root will be the Quotient: For if $a\,b$ be divided by $\sqrt{:}\,a\,b$, the Quotient muſt be $\sqrt{:}\,a\,b$: And if 50 be divided by $\sqrt{:}\,50$, the Quotient will alſo be $\sqrt{:}\,50$. Alſo if any Rational Quantity be to be divided by a Surd, that Rational Quantity muſt firſt be reduced to the Form of a Surd by *Rule* 1.

III. When a Surd Root having a Rational Quantity prefix'd before it, is to be divided by the Surd Part of it, the Quotient will be the Rational Quantity. Thus, If $5\sqrt{:}\,9$, be to be divided by $\sqrt{:}\,9$, the Quotient muſt be 5: As if $5\sqrt{:}\,9$ had been divided by 5, the Quotient would be $\sqrt{:}\,9$.

IV. When the Dividend and Diviſor are the Products of two Rational Quantities multiplied ſeverally into one common Surd; or when they are Rational Quantities prefix'd before one common Surd; then divide the Rational part of the Dividend by the Rational part of the Diviſor; and what reſults is the true Quotient. Thus, if $8\sqrt{:}\,5$ be divided by $2\sqrt{:}\,5$, the Quotient will be 4, and if $8\sqrt[3]{:}\,7$ be divided by $4\sqrt[3]{:}\,7$, the Quotient will be only 2.

But when the Dividend and Diviſor are two Rational Quantities or Numbers prefix'd to two unequal Surds; then you muſt divide, not only as before, the Rational Part of the Dividend by *that* of the Diviſor, but alſo the Surd part; and thoſe

those two Quotients connected together, so as the Rational part stand on the Left-hand, are the true Quotient sought. Thus, If $4\sqrt{:15}$ were to be divided by $2\sqrt{:5}$, the Quotient will be $2\sqrt{:3}$ ($=\sqrt{:12}$) and if $4\sqrt{:12}$, were to be divided by $3\sqrt{:2}$, the Quotient will be $\frac{4}{3}\sqrt{:6}$.

Addition and Subtraction of Surd Roots.

I. WHen two or more Simple and Equal Surds are to be added, Multiply one of them by the Number of them all, and the Product is the Sum required. Thus the Sum of $\sqrt{:5}$, and $\sqrt{:5}$ is the $\sqrt{:20}$; because $\sqrt{:5}$ multiplied by 2, the Number of the Surds, that is by $\sqrt{:4}$, gives $\sqrt{20}$ to their Sum. Also the Sum of $\sqrt[3]{:7}+\sqrt[3]{:7}+\sqrt[3]{:7}$ because the Surds are 3 in Number, is $\sqrt[3]{:189}$; because $\sqrt[3]{7}$ multiplied by 3 (*i. e.*) the $\sqrt[3]{:}$ of 27 makes $\sqrt{:189}$.

II. But if *Unequal* Simple Surds of the same kind are to be added together, or if one be to be subtracted from the other, you must first try whether they are Commensurable; and if they be, that is, if when they have been divided by their greatest

greateſt common Diviſor, their Quotients are Rational Quantities, then you muſt multiply the Sum of thoſe Rational Quantities by the ſaid common Diviſor, and the Product will be the Sum of the Surds propoſed: Or if the Difference of thoſe Rational Quotients be multiplied by the Common Diviſor, then the Product will be the Difference of the given Surds, when the leſs is taken from the greater.

Thus if the *Sum* or *Difference* of theſe two Surds $\surd:50$ and $\surd:8$, were required; becauſe they are unequal, I try firſt, whether they are Commenſurable or not, by dividing each by the greateſt common Diviſor $\surd:2$: And the Quotients are $\surd:25$ and $\surd:4$, that is 5 and 2, which are Rational Numbers; and therefore the Surds are Commenſurable: Then their *Sum* 7, or their *Difference* 3, multiplied by the common Diviſor $\surd:2$, produces $7\surd:2$ for the Sum, and $3\surd:2$ for the *Difference* of the Surds required.

III. If the Commenſurable Surds propoſed, had been Fractions, or Mixt Numbers reduced to the Form of Fractions; they muſt (if they have not one) be reduced to a common Denominator in the leaſt Terms; and then to find out the Rational Quotients, you need only divide the two New Numerators by their greateſt common Diviſor; and then you muſt go on as above, in Integral Surds.

Thus if the *Sum* and *Difference* of $\surd:\frac{24}{25}$ and $\surd:\frac{2}{3}$ were required: When reduced to a com-

mon

mon Denominator, they will be $\sqrt{}:\frac{72}{75}$ and $\sqrt{}:\frac{50}{75}$, and these divided by their greatest common Divisor $\sqrt{}:\frac{2}{75}$, the Quotes are $\sqrt{}:\frac{36}{75}$ and $\sqrt{}:\frac{25}{75}$, or $6\sqrt{}:\frac{2}{75}$ and $5\sqrt{}:\frac{2}{75}$, whose Sum is $\sqrt{}: 11\frac{2}{75}$, and their *Difference* $1\sqrt{}:\frac{2}{75}$.

IV. If the Simple Surds given to be added or subtracted are *Incommensurable*, then they can only (generally speaking) be added or subtracted by the Signs + and —: For neither Sum nor Difference can be express'd by any Single Root. And from this Addition and Subtraction of Simple Surds only by the Signs, arises what they call a *Surd Binomial*, or *Residual* Root. Thus, $\sqrt{}:6 + \sqrt{}:7$, is a *Binomial Surd*, and $\sqrt{}:7 - :\sqrt{6}$ is a *Residual Surd*.

But from *Prop.* 4. *and* 7 *of Euclid's second Book*, there arises a *Rule* which helps us to find the *Sum* or *Difference* of *Incommensurable Square Roots*:

Which Rule is this.

To or from the Sum of the Squares of the given Surd Roots, add or subtract their double Rectangle, and the Square Root of the Sum or Remainder, is the Sum or Difference sought.

E. gr.

E.gr. To find the *Sum* and *Difference* of $\sqrt{:14}$ and $\sqrt{:12}$, their Squares being 14 and 12, their Sum will be 26, and the *Double Rectangle* of $\sqrt{:14}$ into $\sqrt{:12}$, is $2\sqrt{:168}$.

Wherefore $\sqrt{:26 \pm 2\sqrt{:168}}$, is the $\left\{\begin{array}{l}\text{Sum}\\ \text{Difference}\end{array}\right\}$ required.

Of Compound Surds.

THE Arithmetick of *Compound Surds* depends on the Rules above given about *Simple Surds*, and on the true Knowledge of the Signs, and $+$ and $-$ in Algebraick Addition, Subtraction, Multiplication and Divifion; only fome particular Directions may be given as to Binomials and Refiduals: As,

I. If any *Binomial* be to be multiplied by its correfponding Refidual, the Difference of their Squares is the true Product; and therefore will come out a Rational Quantity, as if $\sqrt{:a+e}$, be multiplied by $\sqrt{:a-e}$, the Product will be a Rational Quantity, *viz.* $aa - ee$.

II. Involution in Binomials and Refiduals, is beft and moft eafily performed by a Table of Powers: As becaufe we fee that $aa + 2ae + ee = \square\,\overline{a+e}$. We may conclude that to fquare

any Binomial whatſoever, you need only add the double Rectangle of the Parts to the Sum of the Squares of thoſe Parts; or take the double Rectangle from that Sum, if it be a Reſidual.

III. For Diviſion in *Compound Surds*, 'tis convenient, if not neceſſary, to reduce them firſt to ſome better, and when it can be done, to a Rational Form. And

(1.) If a Binomial conſiſting of two Simple Square Roots, or of one Square Root and Rational Quantity, be multiplied by its correſponding Reſidual, the Product will always be a Rational Quantity.

(2.) If a Binomial conſiſting of two Biquadratick Simple Roots, or of one *ſuch*, and a Rational Quantity; if this be multiplied by its correſponding Reſidual, the Product will be a Reſidual conſiſting of either two Square Roots, or elſe of one Square Root and a Rational Quantity, which Reſidual being multiplied, as is before ſaid, by its Binomial produces a Rational Quantity.

(3.) If a Trinomial having three Simple Square Roots, be multiplied by it ſelf, with one of the Signs changed; the Product will be either a Binomial or Reſidual, which being multiplied by its correſpondent Reſidual or Binomial, will give in the Product a Quantity entirely Rational.

IV. If a Binomial or Reſidual, conſiſting of two Simple Cubick or Biquadratick Roots, &c. or of one

one Cubick or Biquadratick Root; &c. and a Rational Quantity is propoſed for a Diviſor; find ſo many continual Proportionals in the Proportion of the Parts of the Binomial or Reſidual propoſed, as there be Units in the Index of the Radical Sign, and ſuch whoſe Radical Sign may be the ſame with that of the Parts of the Binomial or Reſidual; but conjoyned in the Binomial by + and in Proportionals by + and — alternately; or contrarily, in the Proportionals by +, and in the Reſidual by + and —; the Product of the ſaid Proportionals ſo connexed multiplied into the Binomial or Reſidual, will be a Quantity entirely Rational. After the ſame manner may a Binomial or Reſidual, having 5 or 6, &c. for the Index of the common Radical Sign of the Roots, be reduced to a Quantity entirely Rational.

And Note, *That when the Roots are of different kinds, they muſt firſt be reduced to a common Radical Sign.*

V. If the Diviſor be a Simple Quantity, divide each part of the Dividend by the Diviſor, and connect thoſe particular Products together by their Signs; but if the Diviſor be a Binomial, Trinomial, or Quadrinomial, &c. of ſuch kind as before is ſpecified, reduce that given Diviſor to a new Diviſor that may be a Simple Rational Quantity. Reduce alſo the given Dividend to a new Dividend, by multiplying the former by the Quantities that were Multiplicators, in reducing the given Diviſor to a Rational Quantity; then divide the new Dividend by the new Diviſor: But when the Diviſor cannot be reduced to a

Simple Rational Quantity, ſet the Dividend as a Numerator, over the Diviſor as a Denominator.

Thus, $12 + \surd:63$ divided by 3, the Quotient is $4 + \surd:7$; and $8 - \surd:12$ divided by 2, the Quotient is $4 - \surd:3$; $\surd:21 + \surd:15$ divided by $\surd:3$, the Quotient is $\surd:7 + \surd:5$; $\surd:56 + \surd:24$ divided by $\surd:6$, the Quotient is $\surd:9\frac{2}{6} + 2$: and $\sqrt[3]{}:28 - \sqrt[3]{}:14$ divided by $\sqrt[3]{}:7$, the Quotient is $\sqrt[3]{}:4 - \sqrt[3]{}:2$.

Of

OF
FLUXIONS.

THERE being nothing Publiſh'd on this Subject in our own Language, and yet the vaſt Uſe of this *Method of Inveſtigation*, being as conſpicuous as it is wonderful: I thought it proper to give a ſhort account of it here.

By the *Doctrine of Fluxions* then we are to underſtand the Arithmetick of the *Infinitely ſmall* Increments or Decrements of *Indeterminate* or *Variable Quantities*, or as ſome call them the *Moments*, or *Infinitely ſmall Differences* of ſuch Variable Quantities. Theſe Infinitely ſmall Increments or Decrements, our Incomparable Sir *Iſaac Newton* calls very properly by this Name of *Fluxions*: For as Indeterminate and Variable Quantities, *viz.* ſuch as in the Generation of Curvilineal and other Figures by Local Motion, are continually Increaſing or Diminiſhing, he rightly denominates, *Flowing Quantities*; as being ſuch as are perpetually augmented or leſſen'd by the Flux or Motion of a *Line*, *Surface*, &c. So he calls the *Celerity* or *Velocity* of the Augmentation or Diminution of theſe *Flowing Quantities*, by the Name of *Fluxions*. And becauſe all Figures may be conceived

ceived to be generated by Local Motion ; as is now very commonly ſuppoſed among Geometers: Therefore 'tis much more Natural to conceive the Infinitely ſmall Increments or Decrements of the Variable and *Flowing Quantities*, under the Notion of *Fluxions*, than under that of *Moments* or Infinitely ſmall *Differences*; as *Leibnits*, *Niewentiit*, and the Noble Author of *Analyſe des Infiniment Petits* chuſe rather to take them: Tho' even that Way alſo is not without its Uſe in many Caſes.

The Excellent Sir *Iſ. Newton* ſuppoſes the *Abſciſſa* of a Curve, or any other Flowing or Variable Quantity to be uniformly augmented, and therefore for its *Fluxion* he puts 1, or Unity. And the other Flowing Quantities he denotes uſually by the Letters v, x, y, z, and expreſſes their *Fluxions* by only repeating the ſame Letters with Points, or Pricks over their Heads; thus, $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$: Which are the Fluxions of the former Flowing Quantities. And this Method is much more natural and ſhorter than *Niewentiit*'s, or the *French* one with the Differential d multiplied into the Flowing Quantity, to denote the Fluxion.

And becauſe theſe *Fluxions* themſelves are alſo Indeterminate and Variable Quantities, and do continually increaſe or decreaſe, or grow greater or leſſer: Therefore he conſiders the Velocities with which they do ſo increaſe or diminiſh, as the *Fluxions* of the former *Fluxions*: And thoſe may be called *Second Fluxions*, and are noted with two Points over them, thus, $\ddot{y}$, $\ddot{x}$, $\ddot{z}$. And if you go on again, and conſider the perpetual Augmentation or Diminution of *theſe*, as their *Fluxions* alſo,

alſo, you may make third, fourth, or fifth *Fluxions*, &c. which will be noted thus; $\dddot{y}, \dddot{x}, \dddot{z}$; $\ddddot{y}, \ddddot{x}, \ddddot{z}$; $\overset{\cdot\cdot\cdot\cdot\cdot}{y}, \overset{\cdot\cdot\cdot\cdot\cdot}{x}, \overset{\cdot\cdot\cdot\cdot\cdot}{z}$: And ſo on *ad Infinitum*. If the Flowing Quantity be a Surd or a Fraction, he thus expreſſes its *Fluxion*: Let the Surd be $\sqrt{:a-b}$, its Fluxion is $\sqrt{:\dot{a}-\dot{b}}:$ And the Fluxion of the Fraction $\frac{xx}{d-y}$ is $\frac{xx}{d-\dot{y}}$. See Dr. *Wallis*'s *Latin Edit. Pag.* 392.

The main Buſineſs of the *Algorithm*, or *Arithmetick of Fluxions*, conſiſts in theſe two Things.

I. *From the Flowing Quantity given, to find the Fluxion.*

II. *From the Fluxion to find the Flowing Quantity.*

As to the former of theſe, the Learned Dr. *Wallis*, in the place above mentioned, (from Sir *Iſ. Newton*'s Papers) gives this General Rule.

Let each Term of the Equation be multiplied ſeparately by the ſeveral Indexes of the Powers of all the Flowing Quantities contained in that Term: And in every ſuch Multiplication let one Root or Letter of the Power be changed into its proper Fluxion: So ſhall the Aggregate of all the Products connected together by their proper Signs, be the Fluxion of the Equation deſired.

And

And all the Cafes of it are demonftrated by Sir *Ifaac Newton* in the *Lemmata* above delivered, which I fhall exemplifie by particular Inftances.

I. In the General, to exprefs the Fluxions of Simple Variable Quantities, as was faid before, you need only ufe the Letter or Letters which exprefs them, with a fmall Point over their Heads, thus. The Fluxion of x is $\dot{x}$, and the Fluxion of y is $\dot{y}$, and the Fluxion of $x + y + v + z$, is $\dot{x} + \dot{y} + \dot{v} + \dot{z}$, &c.

And (Inverfely) the *Flowing Quantities* in this Cafe will be eafily had from the Fluxions, by only writing the Letters without the Points over them.

N. B. *For the Fluxion of Permanent Quantities, when any fuch are in the Equation, you muft imagine* o *or a Cypher; for fuch Quantities can have no Fluxions properly fpeaking, becaufe they are without Motion, or Invariable.*

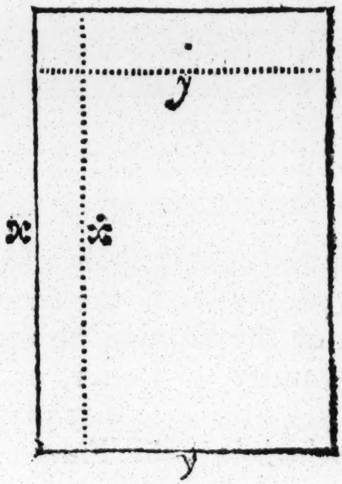
II. To find the Fluxions of the Products of two or more Variable or Flowing Quantities: *Multiply the Fluxion of each Simple Quantity, by the Factors of the Products, or the Product of all the reft, and connect the laft Products by their proper Signs, the Sum, or Aggregate is the Fluxion fought.*

Thus

Thus the Fluxion of xy is $x\dot{y} + \dot{y}x$; and the Fluxion of xyz is $xy\dot{z} + x\dot{y}z + \dot{x}yz$; and the Fluxion of $xvyz$, is $xvy\dot{z} + xv\dot{y}z$; $+ x\dot{v}yz + \dot{x}vyz$, and the Fluxion of $\overline{a + x} \times$ by $b - y$ (the common Product being $ab + bx - ya - xy$) will be $b\dot{x} - \dot{y}a - \dot{x}y - x\dot{y}$.

Demonſtration of Rule 2.

Suppoſe $xy =$ to any Rectangle made or encreaſed by a Perpetual Motion or Fluxions of either of the Sides x or y, along the other, and let the Moments, or Fluxions of the Sides be $\dot{x}$ and $\dot{y}$. By which we underſtand the *Velocity* with which either Side moves to form the Rectangle,



From the Sides ſubduct the half Fluxions of thoſe Sides, and it will ſtand thus; $x - \frac{\dot{x}}{2}$ and $y - \frac{\dot{y}}{2}$.

Multiply theſe one into another; and the Product will be $xy - \frac{y\dot{x}}{2} - \frac{x\dot{y}}{2} + \frac{\dot{x}\dot{y}}{4}$.

Then to the Sides add the half Moments, or Fluxions, and it will be thus: $x + \frac{\dot{x}}{2}$, and $y + \frac{\dot{y}}{2}$: Which multiplied alſo into one another, will produce $xy + \frac{y\dot{x}}{2} + \frac{x\dot{y}}{2} + \frac{\dot{x}\dot{y}}{4}$.

After which, ſubtract the former Product from this laſt, and the Difference will be only $x\dot{y} + \dot{x}y$, the Fluxion of xy, according to the Rule.

The Inverſe of this Rule finds alſo (in this caſe) the *Flowing Quantity* from the *Fluxion*, *viz.* If each Member of the Fluxion be divided by the Fluxionary Quantity, or Letter, or by changing the Fluxionary Letter into that proper Flowing Quantity of which it is the Fluxion. For then the Quotes connected by their proper Signs, will be the Flowing Quantities ſought. Only if the Letters

Letters be all exactly the ſame, the Flowing Quantity will be a Simple one, whoſe Parts are not to be connected together by the Signs + and —, as in the firſt three Examples of this Rule.

III. To find the Fluxion of any Fraction. *Multiply the Fluxion of the Numerator by the Denominator, and after it place (with the Sign —) the Fluxion of the Denominator into the Numerator; and divide the whole by the Square of the Denominator.*

Thus the Fluxion of $\frac{x}{y}$ is $\frac{\dot{x}y - x\dot{y}}{yy}$, for ſuppoſe $\frac{x}{y} = z$: Then will $x = yz$; which equal Quantities will have equal Fluxions; therefore $\dot{x} = \dot{y}z + \dot{z}y$, and $\dot{x} - z\dot{y} = \dot{z}y$; and dividing all by $\dot{y}$: $\frac{\dot{x} - zy}{y} = \dot{z} =$ (becauſe $\frac{x}{y} = z$) $\frac{y\dot{x} - x\dot{y}}{yy}$ wherefore this laſt is the Fluxion of the Fraction $\frac{x}{y} = z$; becauſe z being $= \frac{x}{y}$, $\dot{z}$ will be equal to the Fluxion of $\frac{x}{y}$. And the Fluxion of $\frac{a}{x}$ will be $\frac{-\dot{x}a}{xx}$; for the Permanent Quantity a having no Fluxion, there can be no Product of the Fluxion of the Numerator into the Denominator, as there would have been, had a been x, z, or any other Variable Quantity.

Also the Fluxion of $\frac{x}{a}$ will be $\frac{\dot{x}\dot{a}}{aa}$; and the Fluxion of $\frac{x}{a+x}$ will be $\frac{\dot{x}a + \dot{x}x - x\dot{x}}{aa+2ax+xx}$, or, $\frac{\dot{x}a}{aa+2ax+xx}$; because $-\dot{x}x$ and $+\dot{x}x$ destroy one another: Also the Fluxion of $\frac{1}{x}$, or $x-1$, is $\left(\frac{-\dot{x}}{x^2}\right)$ Here also the Reverse of the Rule serves to find the Flowing Quantity from the Fluxion; as if the Flowing Quantity of this Fraction were required $\frac{\dot{x}y - x\dot{y}}{yy}$: First multiply it by the Square of the Denominator, and it will be $\dot{x}y - x\dot{y}$, from which take away $-x\dot{y}$ which was placed after it, and it will be $\dot{x}y$; omit the Point, and 'tis xy, which because y is the Denominator of a Fraction will at last be $\frac{x}{y}$.

Before the Fluxions of any Power, whether Perfect or Imperfect, can be found; the following Way of Notation must be well understood.

If a Series of Geometrick Progressionals be in this Order,

$1.\ x.\ xx.\ xxx.\ x^4.\ x^5.\ x^6.\ x^7$, &c.

Their

Their Indexes, or Exponents, will be in Arithmetical Progression, and stand thus.

0. 1. 2. 3. 4. 5. 6. 7.

But if they are Fractions; as,

$$\frac{1}{x} \quad \frac{1}{xx} \quad \frac{1}{x^3} \quad \frac{1}{x^4} \quad \frac{1}{x^5} \quad \frac{1}{x^6} \quad \frac{1}{x^7}$$

Then their Exponents will be Negative; and stand thus.

$-1. -2. -3. -4. -5. -6. -7.$

For if you suppose $x = 2$: Then will $\frac{1}{x} = \frac{1}{2}$, and $\frac{1}{xx} = \frac{1}{4}$, and $\frac{1}{xxx} = \frac{1}{8}$, &c.

Or if you express the *Geometrical Series* by means of the Exponent, it will stand thus, x^{-1} x^{-2}, &c. And if it were expressed thus x^0; then it will be $x^0 = 1$, because x is the Denominator of the *Ratio*, in which Unity is not affected. Thus also, $\frac{1}{x^4} = x^{-4}$ and $\frac{1}{x^3} = x^{-3}$. And $1 = x^0$, $x^1 = x$, $x^2 = x\,x$, &c.

Also the Exponent of $\sqrt{:x}$ will be $\frac{1}{2}$, because as $\sqrt{:x}$ is a mean Proportional between 1 and x: So $\frac{1}{2}$ is an Arithmetical Mean between 0 and 1.

And

And the Exponent of $\sqrt[3]{:x}$ will be $\frac{1}{3}$; because as $\sqrt[3]{:x}$ is the first of the two Mean Proportionals between 1 and x; so $\frac{1}{3}$ is the first of the two Arithmetical Means between 0 and 1.

For since $1\ \overset{2}{x}.\ \overset{4}{x\,x}.\ \overset{8}{x\,x\,x}.$ are continually proportional; therefore their Cubes, or any other Roots, will be also continually proportional. That is, $\sqrt[3]{:1}\ (=1.)\ \sqrt[3]{x}.\ \sqrt[3]{x\,x}.\ \sqrt[3]{x\,x\,x}\ (=x)$ ∺

So also,

$1 \because x,\ x^2.\ x\,x\,x.\ x^4.\ x^5.$ ∺.

Wherefore the Roots of the fifth Power of those Quantities will be ∺.

That is,

$\sqrt[5]{1}.\ \sqrt[5]{x}.\ \sqrt[5]{x^2}.\ \sqrt[5]{x^3}.\ \sqrt[5]{x^4}.\ \sqrt[5]{x^5}.\ (= x.)$

Also, for the same Reason, the Exponent of $\sqrt[5]{:x^4}$, will be $\frac{4}{5}$.

N. B. Always place the Index of the Letter (or Power) over that of the Radical Sign.

Thus

Thus in Fractions: The Exponent of $\frac{1}{x}$ will be -1, of $\frac{1}{\sqrt{:x^3}}$ will be $-\frac{3}{2}$, of $\frac{1}{\sqrt[3]{:x^5}}$ will be $-\frac{5}{3}$, of $\frac{1}{\sqrt{:x^7}}$ will be $-\frac{7}{2}$, &c.

N. B. $\sqrt{:x}$ and $x^{\frac{1}{2}}$, or $\sqrt[3]{:x}$ and $x^{\frac{1}{3}}$: Or, $\sqrt[5]{:x^4}$, and $x^{\frac{4}{5}}$, are only two different ways of Notation for one and the same thing, the former is the *old*, the latter the *new* Way.

So likewise $\frac{1}{x}$ and x^{-1} are all one: And $\frac{3}{x}$ is x^{-3}, &c.

The way of reading or expressing Quantities so denoted, is thus, x^{-3} is Unity divided by the Cube of x, and if it were $x^{-\frac{7}{3}}$; it must be read Unity or One divided by the Cube-Root of the Seventh Power of x.

Note also, That the Sum of any two Exponents of two Numbers or Quantities, in Geometrick Progression, makes the Exponent of the Product of those two Terms.

Thus $x^{\frac{1}{2}+\frac{1}{3}}$ or $x^{\frac{5}{6}}$ is the way of expressing the Product of $x^{\frac{1}{3}}$ into $x^{\frac{1}{2}}$, and $x^{-\frac{1}{3}+\frac{1}{5}}$, or $x^{-\frac{2}{15}}$ is the Product of $x^{-\frac{1}{3}}$ into $x^{\frac{1}{5}}$.

Also

Alſo $x^{-\frac{1}{3}-\frac{1}{3}}$ or $x^{-\frac{2}{3}}$ is the Product of $x^{-\frac{1}{3}}$ into itſelf, or the Square of $x^{-\frac{1}{3}}$.

And the Difference between the Exponents of any two Terms, is the Exponent of the Quotient ariſing by Diviſion of the greater by the leſs.

Thus, $x^{\frac{1}{2}-\frac{1}{3}}$ —, or $x^{\frac{1}{6}}$ is the Exponent of the Quotient of $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$, &c.

Let p repreſent the Exponent of N, any Number at pleaſure; and let $p = 1$.

Then will $N^p = N^1 : N^{p+1} = N^2$, and $N^{p+2} = N^3 : N^{p+3} = N^4$, *&c.*

Or, if $p = 3$:

Then will $N^p = N^3$ and $N^{p+3} = N^6$, *&c.*

And Negatively,

$N^{-p} = N^{-3}$, and $N^{-p+3} = N^0$, *&c.*

Alſo o is an Arithmetical Mean between a Poſitive and a Negative Quantity equally diſtant from it, (*i. e.*) —6 : o : 6, are Arithmetically proportional: So is 1 a Geometrical Mean between an Affirmative and Negative Power, at equal Diſtances from it.

That

That is, $N^{-p} : 1 : N^{p} \div\!\div$

Wherefore, $1 = N^{-p} \times N^{p}$. And dividing all by N^{p} : $N^{\frac{1}{p}} = N^{-p}$.

And to add ſome Examples of Multiplication and Diviſion in this way :

$$\frac{1}{x} \times \frac{1}{\sqrt[3]{x^{5}}} = x^{-1} \times x^{\frac{5}{3}} = -\tfrac{3}{3} \times x^{\frac{5}{3}} = x - \tfrac{8}{3}$$

$$= \frac{-1}{x^{\frac{8}{3}}} = \frac{1}{\sqrt[3]{x^{8}}}, \text{ \&c.}$$

And $\frac{1}{\sqrt[3]{x^{5}}}$ divided by $\frac{1}{x}$, will ſtand in this Notation thus, $\frac{1}{x}\Big)\frac{1}{\sqrt[3]{x^{5}}}$, $(= x^{-1})\; x - \tfrac{5}{3} \left(= x - \tfrac{3}{3}\right)$

$$x - \tfrac{5}{3} = (-\tfrac{2}{3} = \frac{1}{\sqrt[3]{x\,x}}, \text{ \&c.}$$

IV. This being well underſtood, there is this general Rule for the finding the Fluxion of any Power, whether Perfect or Imperfect, *viz. Multiply the Power (firſt brought one Degree lower) by the Index of that firſt Power ; and then, that Product by the Fluxion of the Root.*

Thus the Fluxion of xx will be $2x\dot{x}$, for $xx = \dot{x} \times x$, but the Fluxion of $x \times x = \dot{x}x + x\dot{x} = 2\dot{x}x$, &c. and the Fluxion of x^3 will be $3xx\dot{x}$: That of x^3 will be $8x^7\dot{x}$, &c. or if m expreſs the Index of any Power, as ſuppoſe x^m. Then its Fluxion will be $mx^{m+1}\dot{x}$ or $m\dot{x}x^{m-1}$: For x^m brought one degree lower (*m* being a general Index) muſt be x^{m-1}: Then that multiplied by m the Index, makes mx^{m-1}, and this laſt by the Fluxion of the Root produces $mx^{m-1}\dot{x}$.

If the Power be produc'd from a Binomial, *&c.* as ſuppoſe $xx + 2xy + yy$, its Fluxion will be $2xx + 2\dot{x}y + 2\dot{x}y + 2\dot{y}\dot{y}$, by working according to this fourth, and the ſecond Rules.

If the Exponent be Negative; as ſuppoſe x^{-m} or $\frac{1}{x^n}$, its Fluxion will be $-m\dot{x}x^{-m-1}$; or if you would do it by way of Fraction $\frac{-mx^{m-1}}{x^{2m}}\dot{x}$ (for the Square of x^m is as well x^{2m} as x^{m2}) or, according to Mr. *Newton*'s way, which is yet ſhorter, $\frac{-mx}{x^{m+1}}$. See *Caſe* 4. *Page* 252 of his *Principia.*

If

If the Power be imperfect, *i. e.* if its Exponent be a Fraction, as suppose $\sqrt[n]{\ } : x^m$: Or in the other Notation $x^{\frac{m}{n}}$. Let us suppose $x^{\frac{m}{n}} = z$. Then if you raise up each Member to the Power of x, it will stand thus $x^m = z^n$. The Fluxion of which will be, by this general Rule, $m x^{m-1} \dot{x} = n z^{n-1} \dot{z}$. Wherefore $\dot{z}$ will be $= \frac{m \dot{x} x^{m-1}}{n z^{n-1}}$ (by dividing both Parts by $n z^{n-1}$): And

$$\frac{m \dot{x} x^{m-1}}{n z^{n-1}} = \frac{m}{n} x^{\frac{m-n}{n}} \dot{x} \text{: or } \frac{m}{n} \dot{x}^{n} \sqrt{\ } : x^{m-n},$$

by putting instead of $n z^{n-1}$ its Value $n x^{m-\frac{m}{n}}$; So that to find the Fluxion of any kind of Power you must proceed thus,

Multiply the Power given by its Index or Exponent, and then that Product by the Fluxion of the Root of the Power given, and after this subduct one or Unity from the Index of the Power.

As for the Fluxions of Surd Quantities, Mr. *Hayes* gives you many Examples in this Treatise of Fluxions, lately published, which will make the thing plain to any one that will render himself ready at the Practice of this Art. See his Book, *Prop.* VII. *pag.* 14.

U 2 Next

Next for the Rule *Inversly*, to find the Flowing Quantity belonging to the Fluxion of any Power, whether Perfect or Imperfect; proceed thus,

I. Take the Fluxionary Letter or Letters out of the Equation.

II. Augment the Index of the Fluxion by 1, or Unity.

III. Divide the Fluxion by the Index of its Power so increased by Unity.

Examples.

If $x\,x\,\dot{x}$ were proposed: by taking away $\dot{x}$, it will be $3\,x\,x$; and by encreasing its Index by Unity, it will be $3\,x\,x\,x$. Then dividing it by 3, its now (augmented) Index, the Quotient will be $x\,x\,x$, the flowing Quantity required.

Again:

Suppose $\frac{n}{m}\dot{x}x^{\frac{n}{m}-1}$ a Fluxion proposed: By taking away the Fluxionary $\dot{x}$, it will be $\frac{n}{m}x^{\frac{n}{m}-1}$: By augmenting the Index by Unity, (*i.e.*) taking away -1, it will be $\frac{n}{m}\dot{x}^{\frac{n}{m}}$: And lastly, by dividing the remaining Part of the Fluxion by $\frac{n}{m}$ prefixed to, or multiplied into x, the Quotient will be $x^{\frac{n}{m}}$: Which is the Flowing Quantity sought.

You

You will find Examples enough of this *Inverse Method, viz.* The *Calculus Integralis,* or *Summatory Arithmetick,* in Mr. *Hayes*'s Book of Fluxions, *Sect.* 4.

The designed Brevity of this Tract will permit me to give you only two *Uses* of this Doctrine of Fluxions, which, I hope, may serve to give the Inquisitive Reader a whet for a further pursuit of this matter; and he will find sufficient Satisfaction, by perusing the Authors above-mentioned, *viz. Newton, Wallis, Niewentiit, Carre, Leibnitz,* (in the *Act. Eruditor. Lipsiæ*) and especially the Marquiss *L'Hospital,* his excellent *Analyse des Infiniment Petits*: Consult also the Ingenious Mr. *Abraham de Moivre, Specimina Doctrinæ Fluxionum* in *Philosoph. Transact.* N. 216. where you have much in a little on this Subject; and the Marrow of most of these Authors you have in Mr. *Hayes*'s Treatise of Fluxions.

I. To find the Area of a Parabola.

Let the Parameter be $p = 1$, let $x =$ to the *Abscissa,* and $y =$ to the Ordinate.

Then by the Property of the *Curve* $x = yy$ (because $p = 1$.)

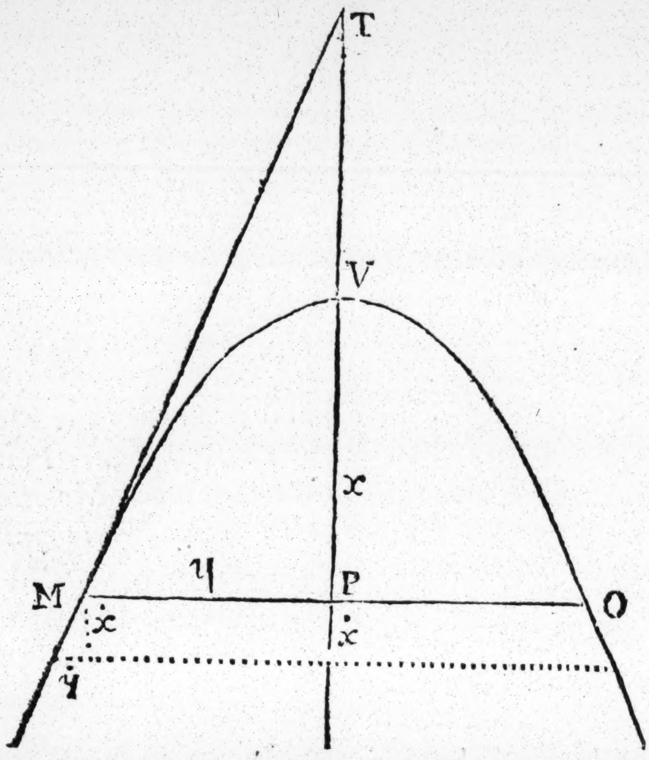
And consequently, by Extraction of the Square Roots of each, and using the new Notation:

$x^{\frac{1}{2}} = y$.

Then multiplying $x^{\frac{1}{2}}$ by $\dot{x}$, the Fluxion of the *Abscissa,* it will stand thus $\dot{x}x^{\frac{1}{2}} =$ to the Fluxion of the Area.

Lastly, Find the flowing Quantity answering to that Fluxion, and that shall give the Area in known Terms.

To



To do which, 1. Take away $\dot{x}$ and it will be $x^{\frac{1}{2}}$.

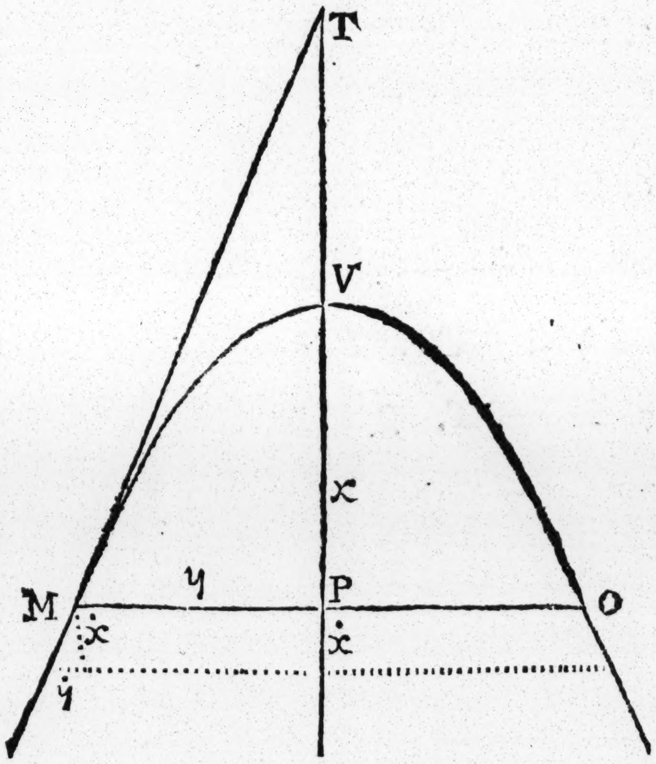
2. Encreasing the Power of $x^{\frac{1}{2}}$ by Unity, it will stand $x\,x^{\frac{1}{2}}$.

3. Divide $x\,x^{\frac{1}{2}}$ by $1 + \frac{1}{2}$, or $\frac{3}{2}$; thus, $\frac{3}{2}\Big)\frac{x\,x^{\frac{1}{2}}}{1}\Big(\frac{2\,x\,x^{\frac{1}{2}}}{3}$: And the Quotient will be $\frac{2\,x\,x^{\frac{1}{2}}}{3}$,

Lastly,

Lastly, Instead of $x^{\frac{1}{2}}$ substitute its Value or Equal $\cancel{p}$: And it will be $\frac{2}{3}\,x\,y =$ to the Area y $m\,V\,p$, and that doubled, gives the whole Area of the Parabola $m\,V\,O$.

II. To find the Point T, where T M being drawn, shall be the true Tangent to the Parabola. The Property of the *Curve* is $p\,x = y\,y$.



1. Find the Fluxion of that Equation, which is $p\,\dot{x} = 2\,y\,\dot{y}$. Wherefore,

2. $\dot{x} = \frac{2\,y\,\dot{y}}{p}$.

3. But

3. But by Similar Triangles, $\dot{y}$, $\dot{x}$, :: y, $\frac{\dot{x}y}{\dot{y}} =$ P T the Sub-tangent.

4. Therefore substituting $\frac{2y\dot{y}}{p}$ (which is $= \dot{x}$, by *Step.* 2.) instead of $\dot{x}$, it will be $\frac{2yy\dot{y}}{\dot{y}p} =$ P T.

5. And then by Expunging the Fluxionary $\dot{y}$, being above and below, it will be $\frac{2yy}{p} =$ P T.

6. Instead of yy, put its Value px, (see the Property of the Curve) it will stand $\frac{2px}{p} =$ P T. That is, by Expunging p.

7. $2x =$ P T. *Q. E. I.*

Wherefore in the Parabola, the Point T is always distant from P, the Foot of the Ordinate, by twice the Length of the *Abscissa.*

FINIS.

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nate,